

The Inverse Inertia Problem for Graphs

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Abstract

Let G be an undirected graph on n vertices and let $\mathcal{S}(G)$ be the set of all real symmetric $n \times n$ matrices whose nonzero off-diagonal entries occur in exactly the positions corresponding to the edges of G . The inverse inertia problem for G asks which inertias can be attained by a matrix in $\mathcal{S}(G)$. We give a complete answer to this question for trees in terms of a new family of graph parameters, the maximal disconnection numbers of a graph. We also give a formula for the inertia set of a graph with a cut vertex in terms of inertia sets of proper subgraphs. Finally, we give an example of a graph that is not inertia-balanced, and investigate restrictions on the inertia set of any graph.

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1 Introduction

Given a simple undirected graph $G = (V, E)$ with vertex set $V = \{1, \dots, n\}$, let $\mathcal{S}(G)$ be the set of all real symmetric $n \times n$ matrices $A = [a_{ij}]$ such that for $i \neq j$, $a_{ij} \neq 0$ if and only if $ij \in E$. There is no condition on the diagonal entries of A .

The set $\mathcal{H}(G)$ is defined in the same way over Hermitian $n \times n$ matrices, and every problem we consider comes in two flavors: the real version, involving $\mathcal{S}(G)$, and the complex version, involving $\mathcal{H}(G)$. There are known examples where a question of the sort we examine here has a different answer when considered over Hermitian matrices rather than over real symmetric matrices [BvdHL1], [Hall], but for each question that is completely resolved in the present paper, the answer over $\mathcal{H}(G)$ proves to be the same as that obtained over $\mathcal{S}(G)$.

The inverse eigenvalue problem for graphs asks: Given a graph G on n vertices and prescribed real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$, is there some $A \in \mathcal{S}(G)$ (or $A \in \mathcal{H}(G)$, alternatively) such that the eigenvalues of A are exactly the numbers prescribed? In general, this is a very difficult problem. Some contributions to its solution appear in [DJ], [JD2], [JS2], [JDS].

A more modest goal is to determine the maximum multiplicity $M(G)$ of an eigenvalue of a matrix in $\mathcal{S}(G)$. This is easily seen to be equivalent to determining the minimum rank $\text{mr}(G)$ of a matrix in $\mathcal{S}(G)$ since $\text{mr}(G) + M(G) = n$. This problem has been intensively studied. Some of the major contributions appear in the papers [F], [N], [JD1], [Hs], [JS1], [vdH], [BFH1], [BvdHL1], [BvdHL2], [BFH2], [BFH3], [BF], [BGL], [JLS], [Hall]. A variant of this problem is the study of $\text{mr}_+(G)$, the minimum rank of all positive semidefinite $A \in \mathcal{S}(G)$. The Hermitian maximum multiplicity $\text{hM}(G)$, Hermitian minimum rank $\text{hmr}(G)$, and Hermitian positive semidefinite rank $\text{hmr}_+(G)$ are defined analogously.

A problem whose level of difficulty lies between the inverse eigenvalue problem and the minimum rank problem for graphs is the inverse inertia

problem, which we now explain.

Definition 1.1. Given a Hermitian $n \times n$ matrix A , the *inertia* of A is the triple

$$(\pi(A), \nu(A), \delta(A)),$$

where $\pi(A)$ is the number of positive eigenvalues of A , $\nu(A)$ is the number of negative eigenvalues of A , and $\delta(A)$ is the multiplicity of the eigenvalue 0 of A . Then $\pi(A) + \nu(A) + \delta(A) = n$ and $\pi(A) + \nu(A) = \text{rank } A$.

If the order of A is also known then the third number of the triple is superfluous. The following definition discards $\delta(A)$.

Definition 1.2. Given a Hermitian matrix A , the *partial inertia* of A is the ordered pair

$$(\pi(A), \nu(A)).$$

We denote the partial inertia of A by $\text{pin}(A)$.

We are interested in the following problem:

Question 1 (Inverse Inertia Problem). Given a graph G on n vertices, for which ordered pairs (r, s) of nonnegative integers with $r + s \leq n$ is there a matrix $A \in \mathcal{S}(G)$ such that $\text{pin}(A) = (r, s)$?

The Hermitian Inverse Inertia Problem is the same question with $\mathcal{H}(G)$ in the place of $\mathcal{S}(G)$. It is well known [JD2, p. 8] that in the case of a tree T most questions over $\mathcal{H}(T)$ are equivalent to questions over $\mathcal{S}(T)$, and in particular if F is a forest and $A \in \mathcal{H}(F)$, then there exists a diagonal matrix D with diagonal entries from the unit circle such that $DAD^{-1} = DAD^* \in \mathcal{S}(F)$. In those sections concerned with the inverse inertia problem for trees and forests we thus assume without loss of generality that every matrix in $\mathcal{H}(F)$ is in fact in $\mathcal{S}(F)$.

In this paper we give a complete solution to the inverse inertia problem for trees and forests. The statement of our solution is a converse to an easier pair of lemmas that apply not just to forests but to any graph.

Lemma 1.1 (Northeast Lemma). *Let G be a graph and suppose that $A \in \mathcal{H}(G)$ with $\text{pin}(A) = (\pi, \nu)$. Then for every pair of integers $r \geq \pi$ and $s \geq \nu$ satisfying $r + s \leq n$, there exists a matrix $B \in \mathcal{H}(G)$ with $\text{pin}(B) = (r, s)$. If in addition A is real, then B can be taken to be real.*

In other words, thinking of partial inertias or Hermitian partial inertias as points in the Cartesian plane, the existence of a partial inertia (π, ν) implies the existence of every partial inertia (r, s) anywhere “northeast” of (π, ν) , as long as $r + s$ does not exceed n . We prove this lemma in Section 2 by perturbing the diagonal entries of A .

To state the second lemma we need to introduce an indexed family of graph parameters.

Definition 1.3. Let G be a graph with n vertices. For any $k \in \{0, \dots, n\}$ we define $\text{MD}_k(G)$, the *maximal disconnection* of G by k vertices, as the maximum, over all induced subgraphs F of G having $n - k$ vertices, of the number of components of F .

For example, $\text{MD}_0(G)$ is the number of components of G , and if T is a tree then $\text{MD}_1(T)$ is the maximum vertex degree of T . Since an induced subgraph cannot have more components than vertices, we always have $k + \text{MD}_k(G) \leq n$.

Remark. As far as we can determine, $\text{MD}_k(G)$ is not a known family of graph parameters. It is, however, related to the toughness t of a graph, which can be defined [C] as

$$t(G) = \min \left\{ \frac{k}{\text{MD}_k(G)} : \text{MD}_k(G) \geq 2 \right\}.$$

For a recent survey of results related to toughness of graphs, see [BBS]. There is also some relation between $\text{MD}_k(G)$ and vertex connectivity: a graph G on n vertices is k -connected, $k < n$, if and only if $\text{MD}_j(G) = 1$ whenever $0 \leq j < k$.

Lemma 1.2 (Stars and Stripes Lemma). *Let G be a graph with n vertices, let $k \in \{0, \dots, n\}$ be such that $\text{MD}_k(G) \geq k$, and choose any pair of integers r and s such that $r \geq k$, $s \geq k$, and $r + s = n - \text{MD}_k(G) + k$. Then there exists a matrix $A \in \mathcal{S}(G)$ such that $\text{pin}(A) = (r, s)$.*

This lemma is proved in Section 2; the idea of the proof is that each partial inertia in the diagonal “stripe” from $(n - \text{MD}_k(G), k)$ northwest to $(k, n - \text{MD}_k(G))$ can be obtained by combining the adjacency matrices of “stars” at each of the k disconnection vertices together with, for each of the remaining components, a matrix of co-rank 1 and otherwise arbitrary inertia.

These two lemmas provide a partial solution to the Inverse Inertia Problem for any graph. Our main result for trees and forests is that for such graphs, and exactly such graphs, the partial solution is complete.

Definition 1.4. Let G be a graph on n vertices. Then (r, s) is an *elementary inertia* of G if for some integer k in the range $0 \leq k \leq n$ we have $k \leq r$, $k \leq s$, and $n - \text{MD}_k(G) + k \leq r + s \leq n$.

The elementary inertias of a graph G are exactly those partial inertias that can be obtained from G by first applying the Stars and Stripes Lemma and then applying the Northeast Lemma. The partial solution given by these lemmas is the following: if (r, s) is an elementary inertia of a graph G , then there exists a matrix $A \in \mathcal{S}(G)$ with $\text{pin}(A) = (r, s)$. This is proved as Observation 2.4 in Section 2.

Theorem 1.1. *The Stars and Stripes Lemma and the Northeast Lemma characterize the partial inertias of exactly forests, as follows:*

1. *Let F be a forest, and let $A \in \mathcal{S}(F)$ with $\text{pin}(A) = (r, s)$. Then (r, s) is an elementary inertia of F .*
2. *Conversely, let G be a graph and suppose that for every $A \in \mathcal{S}(G)$, $\text{pin}(A)$ is an elementary inertia of G . Then G is a forest.*

Of course Claim 1 also applies for $A \in \mathcal{H}(F)$, since for F a forest any matrix in $\mathcal{H}(F)$ is diagonally congruent to a matrix in $\mathcal{S}(F)$ having the same partial inertia. Claim 2 of Theorem 1.1 is a corollary to known results, here called Theorem 2.4. We prove Claim 1 of Theorem 1.1 at the end of Section 5.

In Section 4 we show that determining the set of possible inertias of any graph with a cut vertex can be reduced to the problem of determining the possible inertias of graphs on a smaller number of vertices. The formula we obtain is a generalization of the known formula for the minimum rank of a graph with a cut vertex. In Section 5 we describe elementary inertias in terms of certain edge-colorings of subgraphs, and we show that the same cut-vertex formula proven in Section 4 for inertias also holds when applied to the (usually smaller) set of elementary inertias. Applying these parallel formulas inductively to trees and forests then gives us a proof of Claim 1 of Theorem 1.1. In Section 6 we outline an effective procedure for calculating the set of partial inertias of any tree, using the results of Section 3 to justify some simplifications, and we calculate a few examples. In Section 7 we again consider more general graphs, and demonstrate both an infinite family of forbidden inertia patterns, and the first example of a graph that is not inertia-balanced. The concept of an inertia-balanced graph was introduced in [BF], and determining whether a graph is inertia-balanced is a special case of the inverse inertia problem.

Definition 1.5. A Hermitian matrix A is *inertia-balanced* if

$$|\pi(A) - \nu(A)| \leq 1.$$

A graph G is *inertia-balanced* if there is an inertia-balanced $A \in \mathcal{S}(G)$ with $\text{rank } A = \text{mr}(G)$. A graph G is *Hermitian inertia-balanced* if there is an inertia-balanced $A \in \mathcal{H}(G)$ with $\text{rank } A = \text{hmr}(G)$.

Remark. Our formulation, unlike the definition in [BF], is symmetric in allowing $\nu(A) = \pi(A) + 1$. This doubles the set of inertia-balanced matrices

of odd rank, but the two definitions are equivalent when applied to graphs since $A \in \mathcal{S}(G)$ if and only if $-A \in \mathcal{S}(G)$.

Barioli and Fallat [BF] proved that every tree is inertia-balanced. Theorem 1.1, once proved, will imply a slightly stronger result. The intuition for expecting a graph to be inertia-balanced comes from many small examples in which achieving an eigenvalue of high multiplicity appears to become increasingly difficult as the imbalance increases between the number of eigenvalues that are higher and the number that are lower than the target multiple eigenvalue. The behavior observed in these small examples can be stated formally in terms of the following definitions.

Definition 1.6. A set S of ordered pairs of integers is called *symmetric* if whenever $(r, s) \in S$, then $(s, r) \in S$. A symmetric nonempty set S of ordered pairs of nonnegative integers is called a *stripe* if there is some integer m such that $r + s = m$ for every $(r, s) \in S$, and we specify the particular constant sum by saying that S is a *stripe of rank m* . A stripe S is *convex* if the projection $\{r : (r, s) \in S\}$ is a set of consecutive integers.

Example 1.1. The set $\{(2, 2), (2, 3), (3, 2), (3, 4), (4, 3)\}$ is symmetric, the set $\{(6, 0), (3, 3), (0, 6)\}$ is a stripe, and the stripe $\{(4, 2), (3, 3), (2, 4)\}$ is convex.

Observation 1.3. *Given a graph G of order n and an integer m in the range $\text{mr}(G) \leq m \leq n$, the set*

$$\{\text{pin}(A) : A \in \mathcal{S}(G) \text{ and } \text{rank } A = m\}$$

is a stripe of rank m . The same is true for $A \in \mathcal{H}(G)$ with m in the range $\text{hmr}(G) \leq m \leq n$.

Proof. Symmetry comes from the fact that $-A \in \mathcal{S}(G)$ if and only if $A \in \mathcal{S}(G)$, and similarly for $\mathcal{H}(G)$. The sets are nonempty by the definitions of $\text{mr}(G)$ and $\text{hmr}(G)$ and the Northeast Lemma. \square

Definition 1.7. A graph G is *inertia-convex on stripes* or *Hermitian inertia-convex on stripes* if each of the stripes defined in Observation 1.3 (with $A \in \mathcal{S}(G)$ or $A \in \mathcal{H}(G)$, respectively) is convex.

In other words, a graph is inertia-convex on stripes if each stripe of possible partial inertias does not contain a gap.

Corollary 1.2 (Corollary to Theorem 1.1). *Every forest is inertia-convex on stripes.*

Proof. Let F be a forest. By Theorem 1.1, each of the stripes defined in Observation 1.3 is the set of elementary inertias of some fixed rank m . For each fixed k with $\text{MD}_k(F) \geq k$ we obtain a set of elementary inertias which is a union of convex stripes. It follows that for any fixed m , the set of elementary inertias of rank m is the union of convex stripes of rank m as k varies over all allowed integers. Since a union of convex stripes of the same rank is a single convex stripe, each of the stripes defined in Observation 1.3 is convex. \square

It has been an outstanding question if there is any graph that is not inertia-balanced. At the AIM Workshop in Palo Alto in October 2006, the prevailing opinion was that such a graph does not exist [BHS].

In Section 7 we give an example of a graph that is not inertia-balanced. First we show that every graph satisfies a condition that is much weaker than inertia-balanced (except in the case of minimum semidefinite rank 2). The counterexample graph and new condition together allow us to completely determine which sets can occur as the complement of the set of possible partial inertias of a graph G with $\text{mr}_+(G) \leq 3$. The possible excluded partial inertia sets giving minimum semidefinite rank 4 or greater remain unclassified.

For the most part our notation for graphs follows Diestel [D]. We make use specifically of the following notation throughout:

- All graphs are simple, and a graph is formally an ordered pair $G = (V, E)$ where V is a finite set and E consists of pairs from V . When

referring to an individual edge, we abbreviate $\{u, v\}$ to uv or vu . The vertex set of a graph G is also referred to as $V(G)$, and the edge set as $E(G)$.

- For $S \subseteq V$, $G[S]$ is the subgraph of G induced by S and $G - S$ is the induced subgraph on $V(G) \setminus S$. We write $G - F$ rather than $G - V(F)$ and $G - v$ rather than $G - \{v\}$.
- The number of vertices of a graph G is denoted $|G|$.
- K_n is the complete graph on n vertices.
- $S_n = (\{1, 2, 3, \dots, n\}, \{12, 13, \dots, 1n\})$ is called the *star graph* on n vertices. This is the same as the complete bipartite graph $K_{1, n-1}$.
- P_n is the path on n vertices. Paths are described explicitly by concatenating the names of the vertices in order; for example, uvw denotes the graph $(\{u, v, w\}, \{uv, vw\})$.
- If v is a vertex of G , $d(v)$ is the degree of v .
- $\Delta(G) = \max \{d(v) : v \in V(G)\}$.

We conclude the introduction with some elementary facts about inertia, and include short proofs to keep the paper self-contained.

Proposition 1.4. *Let A be a Hermitian $n \times n$ matrix and let B be a principal submatrix of A of size $(n - 1) \times (n - 1)$. Then*

$$\pi(A) - 1 \leq \pi(B) \leq \pi(A) \quad \text{and} \quad \nu(A) - 1 \leq \nu(B) \leq \nu(A).$$

Proof. By the interlacing inequalities [B]

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A and μ_1, \dots, μ_{n-1} are the eigenvalues of B , arranged in decreasing order. If $\mu_1 \leq 0$, $\pi(B) = 0 \leq \pi(A)$. Otherwise,

let $m \in \{1, \dots, n-1\}$ be the largest integer with $\mu_m > 0$. Then $\lambda_m > 0$ and $\pi(B) = m \leq \pi(A)$.

If $\lambda_1 \leq 0$, $\pi(B) = 0 > \pi(A) - 1$. Otherwise, let $\ell \in \{1, \dots, n\}$ be the largest integer with $\lambda_\ell > 0$. Then $\mu_{\ell-1} \geq \lambda_\ell > 0$ and $\pi(B) \geq \ell - 1 = \pi(A) - 1$.

Similarly, $\nu(A) - 1 \leq \nu(B) \leq \nu(A)$. \square

Proposition 1.5 (Subadditivity). *Let A, B be Hermitian $n \times n$ matrices and let $C = A + B$. Then*

$$\pi(C) \leq \pi(A) + \pi(B) \quad \text{and} \quad \nu(C) \leq \nu(A) + \nu(B).$$

Proof. If $\pi(A) + \pi(B) \geq n$, the first inequality is true, so assume that $\pi(A) + \pi(B) < n$. Let $i = \pi(A)$ and $j = \pi(B)$. Then $\lambda_{i+1}(A) \leq 0$ and $\lambda_{j+1}(B) \leq 0$. By the Weyl inequalities [B],

$$\lambda_{i+j+1}(C) = \lambda_{i+1+j+1-1}(C) \leq \lambda_{i+1}(A) + \lambda_{j+1}(B) \leq 0.$$

Therefore $\pi(C) \leq i + j = \pi(A) + \pi(B)$.

Similarly, $\nu(C) \leq \nu(A) + \nu(B)$. \square

Proposition 1.6. *Let A be a Hermitian $n \times n$ matrix and let $cx x^*$ be a Hermitian rank 1 matrix (so c is real-valued). Then*

$$\pi(A + cx x^*) \leq \begin{cases} \pi(A) + 1 & \text{if } c > 0 \\ \pi(A) & \text{if } c < 0 \end{cases}$$

and

$$\nu(A + cx x^*) \leq \begin{cases} \nu(A) + 1 & \text{if } c < 0 \\ \nu(A) & \text{if } c > 0 \end{cases}$$

Proof. Let $c > 0$. Then $\pi(cx x^*) = 1$, $\nu(cx x^*) = 0$. By Proposition 1.5,

$$\begin{aligned} \pi(A + cx x^*) &\leq \pi(A) + \pi(cx x^*) = \pi(A) + 1, \\ \nu(A + cx x^*) &\leq \nu(A) + \nu(cx x^*) = \nu(A). \end{aligned}$$

The argument is similar if $c < 0$. \square

2 The inertia set of a graph

Definition 2.1. Let \mathbb{N} be the set of nonnegative integers, and let $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. We define the following sets:

$$\begin{aligned}\mathbb{N}_{\leq k}^2 &= \{(r, s) \in \mathbb{N}^2 : r + s \leq k\}, \\ \mathbb{N}_{\geq k}^2 &= \{(r, s) \in \mathbb{N}^2 : r + s \geq k\}, \\ \mathbb{N}_{[i, j]}^2 &= \mathbb{N}_{\geq i}^2 \cap \mathbb{N}_{\leq j}^2, \\ \mathbb{N}_i^2 &= \mathbb{N}_{[i, i]}^2 \text{ (the complete stripe of rank } i\text{).}\end{aligned}$$

We note that a stripe of rank i is a nonempty symmetric subset of \mathbb{N}_i^2 .

Definition 2.2. Given a graph G , we define

$$\mathcal{I}(G) = \{(r, s) : \text{pin}(A) = (r, s) \text{ for some } A \in \mathcal{S}(G)\},$$

and

$$\text{h}\mathcal{I}(G) = \{(r, s) : \text{pin}(A) = (r, s) \text{ for some } A \in \mathcal{H}(G)\}.$$

We call $\mathcal{I}(G)$ the *inertia set* of G and $\text{h}\mathcal{I}(G)$ the *Hermitian inertia set* of G .

Now suppose $(r, s) \in \mathcal{I}(G)$ and let $A \in \mathcal{S}(G)$ with $\text{pin}(A) = (r, s)$. Since $r + s = \text{rank } A$, we have $\text{mr}(G) \leq r + s \leq |G|$. We record this as

Observation 2.1. *Given a graph G on n vertices, $\mathcal{I}(G) \subseteq \mathbb{N}_{[\text{mr}(G), n]}^2$ and $\text{h}\mathcal{I}(G) \subseteq \mathbb{N}_{[\text{hmr}(G), n]}^2$.*

The fact that every real symmetric matrix is also Hermitian immediately gives us:

Observation 2.2. *For any graph G , $\mathcal{I}(G) \subseteq \text{h}\mathcal{I}(G)$ and $\text{hmr}(G) \leq \text{mr}(G)$.*

The Northeast Lemma, as stated in the Introduction, substantially shortens the calculation of the inertia set of a graph.

Proof of Northeast Lemma. Let G be a graph and suppose that $(\pi, \nu) \in \text{h}\mathcal{I}(G)$, and let $(r, s) \in \mathbb{N}_{\leq n}^2$ be given with $r \geq \pi$ and $s \geq \nu$. We wish to show that $(r, s) \in \text{h}\mathcal{I}(G)$. If in addition $(\pi, \nu) \in \mathcal{I}(G)$, we must show that $(r, s) \in \mathcal{I}(G)$.

Let $A \in \mathcal{H}(G)$ with $\text{pin}(A) = (\pi, \nu)$. If $\pi + \nu = n$ there is nothing to prove, so assume $\pi + \nu < n$. It suffices to prove that there exists a $B \in \mathcal{H}(G)$ with $\text{pin}(B) = (\pi + 1, \nu)$, because then an analogous argument can be given to prove that there is a $C \in \mathcal{H}(G)$ with $\text{pin}(C) = (\pi, \nu + 1)$ and these two facts may be applied successively to reach (r, s) . We also need to ensure that when A is real symmetric B is also real symmetric. Choose $\varepsilon > 0$ such that $A + \varepsilon I$ is invertible and $\nu(A + \varepsilon I) = \nu$. Then $\pi(A + \varepsilon I) = n - \nu$. Let $A_0 = A$ and then perturb the diagonal entries in order: for any $i \in \{1, \dots, n\}$ let $A_i = A_{i-1} + \varepsilon e_i e_i^*$, so that $A_n = A + \varepsilon I$. Then $A_i \in \mathcal{H}(G)$ for $i = 0, \dots, n$ and by Propositions 1.5 and 1.6,

$$\pi(A_{i-1}) \leq \pi(A_i) \leq \pi(A_{i-1}) + 1$$

for $i = 1, \dots, n$. It follows that every integer in $\{\pi, \pi + 1, \dots, n - \nu\}$ is equal to $\pi(A_i)$ for some $i \in \{0, \dots, n\}$. Since

$$\nu = \nu(A + \varepsilon I) \leq \nu(A_{n-1}) \leq \dots \leq \nu(A_2) \leq \nu(A_1) \leq \nu(A) = \nu$$

by Proposition 1.6, $\nu(A_i) = \nu$ for $i = 0, 1, \dots, n$. Then for some i we have $\text{pin}(A_i) = (\pi + 1, \nu)$, and we can take $B = A_i$. As desired, B is real symmetric if A is real symmetric, which completes the $\mathcal{S}(G)$ version of the Northeast Lemma as well as the $\mathcal{H}(G)$ version: Within either one of the two inertia sets $\mathcal{I}(G)$ or $\text{h}\mathcal{I}(G)$, the existence of a partial inertia (π, ν) implies the existence of every partial inertia (r, s) within the triangle

$$r \geq \pi, \quad s \geq \nu, \quad r + s \leq n,$$

or in other words every partial inertia to the “northeast” of (π, ν) . \square

Definition 2.3. If a graph G on n vertices satisfies $\mathcal{I}(G) = \mathbb{N}_{[\text{mr}(G), n]}^2$ we say that G is *inertially arbitrary*. If a graph G on n vertices satisfies $\text{h}\mathcal{I}(G) = \mathbb{N}_{[\text{hmr}(G), n]}^2$ we say that G is *Hermitian inertially arbitrary*.

Example 2.1. The complete graph K_n , $n \geq 2$. Since $\pm J_n$ (the all ones matrix) $\in \mathcal{S}(K_n)$, $(1, 0), (0, 1) \in \mathcal{I}(K_n)$. By the Northeast Lemma $\mathbb{N}_{[1, n]}^2 \subseteq \mathcal{I}(K_n)$. Since $\mathcal{I}(K_n) \subseteq \mathbb{N}_{[\text{mr}(K_n), n]}^2 = \mathbb{N}_{[1, n]}^2$ by Observation 2.1, K_n is inertially arbitrary.

Example 2.2. The path P_n , $n \geq 2$. A consequence of a well-known result of Fiedler [F] is that for a graph G on n vertices, $\text{mr}(G) = n - 1$ if and only if $G = P_n$. It follows from a Theorem in [Hald] that there is an $A \in \mathcal{S}(P_n)$ with eigenvalues $1, 2, 3, \dots, n$. Then for $k = 1, \dots, n$, $\text{pin}(A - kI) = (n - k, k - 1)$. By the Northeast Lemma, $\mathcal{I}(P_n) = \mathbb{N}_{[n-1, n]}^2 = \mathbb{N}_{[\text{mr}(P_n), n]}^2$, so P_n is also inertially arbitrary.

The partial inertia set for a graph on n vertices can never be smaller than the partial inertia set for P_n .

Proposition 2.3. If G is any graph on n vertices, $\mathbb{N}_{[n-1, n]}^2 \subseteq \mathcal{I}(G)$.

Proof. Let $r, s \in \{0, 1, \dots, n\}$ with $r + s = n$. Let

$$D = \text{diag}(r, r - 1, \dots, 2, 1, -1, -2, \dots, -s)$$

and let A_G be the adjacency matrix of G . By Gershgorin's theorem, $B = D + \frac{1}{2n}A_G \in \mathcal{S}(G)$ has eigenvalues $\lambda_1 > \lambda_2 > \dots > \lambda_r > 0 > \lambda_{r+1} > \dots > \lambda_n$, so $\text{pin}(B) = (r, s)$. Furthermore for $r < n$, $B - \lambda_{r+1}I_n \in \mathcal{S}(G)$ has partial inertia $(r, s - 1)$. It follows that $\mathbb{N}_{[n-1, n]}^2 \subseteq \mathcal{I}(G)$. \square

The fact that inertia sets are additive on disconnected unions of graphs (Observation 4.1) gives us an immediate corollary.

Corollary 2.1. If G is any graph on n vertices and G has ℓ components, $\mathbb{N}_{[n-\ell, n]}^2 \subseteq \mathcal{I}(G)$.

The existence of a complete stripe of partial inertias of rank $n - \ell$ plays a role in the proof of our second lemma from the Introduction.

Proof of the Stars and Stripes Lemma. Let G be a graph with n vertices, and let $S \subseteq V(G)$ be such that $|S| = k$ and $G - S$ has $\text{MD}_k(G)$ components, with $\text{MD}_k(G) \geq k$. Also, let (r, s) be any pair of integers such that $k \leq r$, $k \leq s$, and $r + s = n - \text{MD}_k(G) + k$.

Without loss of generality label the vertices of G so that $S = \{1, \dots, k\}$, and for each vertex $1 \leq v \leq k$ let A_v be the $n \times n$ adjacency matrix of the subgraph of G that retains all vertices of G , but only those edges that include the vertex v . If v is isolated in G then $\text{pin}(A_v) = (0, 0)$; otherwise the subgraph is a star plus isolated vertices and $\text{pin}(A_v) = (1, 1)$.

Now $G - S$ is a graph with $n - k$ vertices and $\text{MD}_k(G)$ components, so by Corollary 2.1 there exists a matrix $B \in \mathcal{S}(G - S)$ with $\text{pin}(B) = (r - k, s - k)$. Let C be the direct sum of the $k \times k$ zero matrix with B , so that the rows and columns of C are indexed by the full set $V(G)$, as is the case with the matrices A_1, \dots, A_k . Let $M = A_1 + A_2 + \dots + A_k + C$. Then $M \in \mathcal{S}(G)$, and by subadditivity of partial inertias (Proposition 1.5) we also have $\pi(M) \leq r - k + k = r$ and $\nu(M) \leq s - k + k = s$. Since $r + s \leq n$ we can apply the Northeast Lemma to conclude that $(r, s) \in \mathcal{I}(G)$. \square

As mentioned in the introduction, the partial inertias which can be deduced from Lemmas 1.1 and 1.2 are precisely the elementary inertias.

Definition 2.4. Let G be a graph on n vertices. Then the *set of elementary inertias of G* , $\mathcal{E}(G)$, is given by

$$\mathcal{E}(G) = \{(r, s) \in \mathbb{N}^2 : (r, s) \text{ is an elementary inertia of } G\}.$$

We may also think of $\mathcal{E}(G)$ as follows: For each integer k , $0 \leq k \leq n$, let

$$T_k = \{(x, y) \in \mathbb{R}^2 : k \leq x, k \leq y, n - \text{MD}_k(G) + k \leq x + y \leq n\},$$

and let $T = \bigcup_{k=0}^n T_k$. Each nonempty T_k is a possibly degenerate trapezoid, and

$$\mathcal{E}(G) = \mathbb{N}_{\leq n}^2 \cap T.$$

Observation 2.4. *For any graph G , we have $\mathcal{E}(G) \subseteq \mathcal{I}(G)$.*

Proof. Let G be a graph on n vertices, and suppose $(r, s) \in \mathcal{E}(G)$. Then for some integer k we have

$$k \leq r, \ k \leq s, \text{ and } n - \text{MD}_k(G) + k \leq r + s \leq n.$$

(Note that this implies $\text{MD}_k(G) \geq k$.) Recall that $k + \text{MD}_k(G) \leq n$, so

$$k + k \leq n - \text{MD}_k(G) + k \leq r + s.$$

It follows that there is an ordered pair of integers (x, y) satisfying

$$k \leq x \leq r, \ k \leq y \leq s, \text{ and } x + y = n - \text{MD}_k(G) + k.$$

The Stars and Stripes Lemma gives us $(x, y) \in \mathcal{I}(G)$, after which the North-east Lemma gives us $(r, s) \in \mathcal{I}(G)$ since $r + s \leq n$. \square

Remark. Given a graph F on m vertices there is a smallest integer a such that $\mathbb{N}_a^2 \subseteq \mathcal{I}(F)$. If F is inertia-convex on stripes then a is the same as $\text{mr}_+(F)$, and if F is inertially arbitrary then a is the same as $\text{mr}(F)$. Suppose that F is $G - S$ as in the definition of $\text{MD}_k(G)$, with $|S| = k$ and $m = n - k$. Then some trapezoid of elementary inertias of G comes from the easy estimate that the maximum co-rank of arbitrary inertia for F , i.e. $m - a$, is at least the number of components ℓ of F (Corollary 2.1). Suppose we had an improved lower bound $\Xi(F)$ for this co-rank, a graph parameter that always satisfies $\ell \leq \Xi(F) \leq m - a$. (The improvement $\ell \leq \Xi(F)$ will be guaranteed, for example, if Ξ is additive on the components of F and is at least 1 on each component.) We could then define a family of graph parameters analogous to $\text{MD}_k(G)$ by defining $\text{M}\Xi_k(G)$ to be the maximum, over all subsets $S \subseteq V(G)$

of size $|S| = k$, of $\Xi(G - S)$. Replacing $\text{MD}_k(G)$ by $\text{M}\Xi_k(G)$ would then give a stronger version of the Stars and Stripes Lemma, and an expanded set of not-as-elementary inertias.

For any graph G , the Stars and Stripes Lemma gives us a bound on the maximum eigenvalue multiplicity $\text{M}(G)$.

Corollary 2.2. *Let G be a graph on n vertices. Then for any $0 \leq k \leq n$, $\text{M}(G) \geq \text{MD}_k(G) - k$.*

When this bound is attained, it is attained in particular on a set that includes the center of the stripe $\mathbb{N}_{\text{mr}(G)}^2$.

Corollary 2.3. *Let G be a graph. If $\text{MD}_k(G) - k = \text{M}(G)$ for some k , then G is inertia-balanced.*

Example 2.3. The n -sun H_n is defined as the graph on $2n$ vertices obtained by attaching a pendant vertex to each vertex of an n -cycle [BFH1]. We have $\text{MD}_0(H_n) = 1$ and $\text{MD}_k(H_n) = 2k$ for $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$. It follows that, in addition to $(2n - 1, 0)$ and $(0, 2n - 1)$, $\mathcal{I}(H_n)$ contains every integer point (r, s) within the trapezoid

$$r + s \leq 2n, \quad 2n \leq r + 2s, \quad 2n \leq 2r + s, \quad 3n \leq 2r + 2s.$$

Since for $n > 3$ it is known that $\text{mr}(H_n) = 2n - \lfloor \frac{n}{2} \rfloor$ [BFH1], this shows that the n -sun is inertia-balanced for $n > 3$.

It is useful to note the following connection between the inverse inertia problem and the minimum semidefinite rank problem.

Observation 2.5. *The inertia set of a graph restricted to an axis gives*

$$\begin{aligned} \mathcal{I}(G) \cap (\mathbb{N} \times \{0\}) &= \{(k, 0) : k \in \mathbb{N}, \text{mr}_+(G) \leq k \leq n\}, \\ \mathcal{I}(G) \cap (\{0\} \times \mathbb{N}) &= \{(0, k) : k \in \mathbb{N}, \text{mr}_+(G) \leq k \leq n\}, \end{aligned}$$

and similarly for $\text{h}\mathcal{I}(G)$ and $\text{hmr}_+(G)$.

In other words, solving the inverse inertia problem for a graph G on the x -axis (or y -axis) is equivalent to solving the minimum semidefinite rank problem for G . One well-known result about minimum semidefinite rank is:

Theorem 2.4 (hmr_+ [vdH], mr_+ [BFH3]). *Given a graph G on n vertices, $\text{hmr}_+(G) = n - 1$ if and only if G is a tree, and $\text{mr}_+(G) = n - 1$ if and only if G is a tree.*

As noted in Example 2.2, if G is not P_n then $\text{mr}(G) \neq n - 1$, and therefore $\text{mr}(G) < n - 1$. It follows that $\{P_n\}_{n=1}^\infty$ are the only inertially arbitrary trees.

If G is not connected then any matrix in $\mathcal{S}(G)$ is a direct sum of smaller matrices, which shows that mr_+ is additive on the components of a graph.

Observation 2.6. *Let G be a graph on n vertices and let ℓ be the number of components of G . Then $\text{mr}_+(G) = n - \ell$ if and only if G is a forest.*

This gives us a statement that implies the second claim of Theorem 1.1.

Corollary 2.5. *Let G be a graph. Then $(\text{mr}_+(G), 0)$ is an elementary inertia of G if and only if G is a forest.*

Proof. Let G be a graph on n vertices and let ℓ be the number of components of G . Since $\text{MD}_0(G) = \ell$, $(i, 0)$ is an elementary inertia of G exactly for those integers i in the range $n - \ell \leq i \leq n$. In particular, $(\text{mr}_+(G), 0)$ is an elementary inertia if and only if $\text{mr}_+(G) = n - \ell$. By Observation 2.6, this is true if and only if G is a forest. \square

Although the Stars and Stripes Lemma only gives the correct value of $\text{mr}_+(G)$ when G is a forest, we have already seen that it can give the correct values of $\text{mr}(G)$ and $M(G)$ for some graphs containing a cycle.

Question 2. What is the class of graphs for which

$$M(G) = \max_{0 \leq k \leq n} \{\text{MD}_k(G) - k\}?$$

Theorem 1.1 implies that this class includes all forests, and Example 2.3 shows that the class includes the n -sun graphs H_n for $n > 3$.

Example 2.4. $G = S_n$, $n \geq 4$. Let A be the adjacency matrix of S_n . Then $\text{pin}(A) = (1, 1)$. Since $\text{mr}(S_n) = 2$ and $\text{mr}_+(S_n) = n - 1$, by the Northeast Lemma we have

$$\begin{aligned} \mathcal{I}(S_n) &= \{(n-1, 0), (n, 0), (0, n-1), (0, n)\} \\ &\cup \{(r, s) : r \geq 1, s \geq 1, r + s \leq n\}. \end{aligned}$$

It follows that S_n is not inertially arbitrary.

As has already been noted, if $A \in \mathcal{H}(G)$ with $\text{pin}(A) = (r, s)$, then $-A \in \mathcal{H}(G)$ with $\text{pin}(A) = (s, r)$, and if A is real then $-A$ is real. A consequence of Observation 1.3 is

Observation 2.7 (Symmetry property). *The sets $\mathcal{I}(G)$ and $\text{h}\mathcal{I}(G)$ are symmetric about the line $y = x$.*

3 Principal parameters for the inertia set of a tree

The purpose of this section is to define some basic parameters associated with a tree, establish their fundamental properties, and relate them to the maximal disconnection numbers $\text{MD}_k(G)$. In Section 6 we will use these results to simplify the application of Theorem 1.1.

In [JD1], Johnson and Duarte computed the minimum rank of all matrices in $\mathcal{S}(T)$, where T is an arbitrary tree. One of the graph parameters used by them, the *path cover number* of T , is also needed in our work. It is defined as follows.

Definition 3.1. Let T be a tree.

- (a) A *path cover* of T is a collection of vertex disjoint paths, occurring as induced subgraphs of T , that covers all the vertices of T .

- (b) The *path cover number* of T , $P(T)$, is the minimum number of paths occurring in a path cover of T .
- (c) A *path tree* \mathcal{P} is a path cover of T consisting of $P(T)$ disjoint paths, say $Q_1, Q_2, \dots, Q_{P(T)}$. An *extra edge* is an edge of T that is incident to vertices on two distinct Q 's. Clearly there are exactly $P(T) - 1$ extra edges.

The Theorem of Duarte and Johnson is

Theorem 3.1. *For any tree T on n vertices,*

$$\text{mr}(T) + P(T) = n.$$

As indicated, $P(T)$ will also be used in our work. This is not surprising, because inertia is a refinement of rank. Our use of $P(T)$ will be made precise now. First, we need another definition.

Definition 3.2. Let $G = (V, E)$ be a graph, and let $S \subseteq V$. Let

$$E_G(S) = \{xy \in E : x \in S \text{ or } y \in S\},$$

that is, $E_G(S)$ consists of all edges of G that are incident to at least one vertex in S .

We define now an integer-valued mapping f on the set of all subsets of V by:

$$f_G(S) = |E_G(S)| - 2|S| + 1.$$

Observation 3.1. *For any graph G , $f_G(\emptyset) = 1$.*

Observation 3.2. *Let T be a tree on n vertices, and choose an integer k in the range $0 \leq k \leq n$. Then*

- *For every $S \subseteq V(T)$ with $|S| = k$, $f_T(S) \leq \text{MD}_k(T) - k$, and*
- *For some $S \subseteq V(T)$ with $|S| = k$, $f_T(S) = \text{MD}_k(T) - k$.*

Proof. In any forest, the number of components plus the number of edges equals the number of vertices. Let $S \subseteq V$ with $|S| = k$. The forest $T - S$ has $n - 1 - E_T(S)$ edges and $n - k$ vertices, so it has $E_T(S) - k + 1$ components. By definition of $\text{MD}_k(T)$, $E_T(S) - k + 1 \leq \text{MD}_k(T)$, or equivalently, $f_T(S) \leq \text{MD}_k(T) - k$. Since $E_T(S) - k + 1 = \text{MD}_k(T)$ for some S with $|S| = k$, the second statement follows. \square

Our first theorem in this section is the following:

Theorem 3.2. *Let T be a tree with $|T| = n$ and let $P(T)$ denote its path cover number. Then*

$$P(T) = \max_{S \subseteq V} \{f_T(S)\} = \max_{0 \leq k \leq n} \{\text{MD}_k(T) - k\}.$$

The second equality is a direct consequence of Observation 3.2. The first equality will be proved by induction on $|T|$, but first we prove it directly for several special cases. These special cases will also be used in the proof of Theorem 3.2.

Observation 3.3. *Theorem 3.2 holds for any path.*

Proof. Let P_n denote the path on n vertices, $n \geq 1$. The degree of any vertex in P_n is at most two, so for any $\emptyset \neq S \subseteq V(P_n)$, $f_{P_n}(S) \leq 2|S| - 2|S| + 1 = 1$. The result follows by Observation 3.1. \square

Corollary 3.3. *Theorem 3.2 holds for any tree T with $|T| \leq 3$.*

Observation 3.4. *Theorem 3.2 holds for S_n , for any $n \geq 3$.*

Proof. Label the pendant vertices of S_n by $2, 3, \dots, n$ and the vertex of degree $n - 1$ by 1. It is known that $P(S_n) = n - 2$. For $S = \{1\}$, we have $f_{S_n}(S) = n - 1 - 2 + 1 = n - 2$, while for any $S \subseteq V(S_n)$ it is straightforward to see that $f_{S_n}(S) \leq n - 2$. \square

Lemma 3.5. *Let T be a tree, and let $\emptyset \neq S \subseteq V(T)$. Then $f_T(S) \leq P(T)$.*

Proof. The proof is by induction on $|T|$. Corollary 3.3 covers the base of the induction, so we proceed to the general induction step.

Let \mathcal{P} be any path tree of T , consisting of paths $Q_1, Q_2, \dots, Q_{P(T)}$. There are $P(T) - 1$ extra edges. We can assume without loss of generality that Q_1 is a pendant path in \mathcal{P} (so exactly one extra edge emanates from it), and we denote by v the vertex of Q_1 that is incident to an extra edge.

We can also assume without loss of generality that no vertex of S has degree 1 or 2, since deleting such a vertex cannot increase the value of the function $f_T(S)$ that we are trying to bound from above.

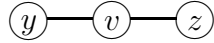
Case 1. v is an end vertex of Q_1 . In this case $S \subseteq \bigcup_{i=2}^{P(T)} V(Q_i)$. Also, $P(T - Q_1) = P(T) - 1$. Applying the induction hypothesis, we get

$$f_T(S) \leq f_{T-Q_1}(S) + 1 \leq P(T - Q_1) + 1 = P(T).$$

Case 2. v is an internal vertex of Q_1 . Suppose first that one of the two end vertices of Q_1 (call it z) is at distance (in Q_1) of at least two from v . Then $P(T - z) = P(T)$ and $S \subseteq V(T - z)$. Moreover, by the induction hypothesis,

$$f_T(S) = f_{T-z}(S) \leq P(T - z) = P(T).$$

Hence we may assume that Q_1 has the form yvz :



We have $P(T - Q_1) = P(T) - 1$. If $v \notin S$ then, by induction,

$$f_T(S) \leq f_{T-Q_1}(S) + 1 \leq P(T - Q_1) + 1 = P(T).$$

If $v \in S$, then

$$f_T(S) = f_{T-Q_1}(S \setminus \{v\}) + 3 - 2 \leq P(T - Q_1) + 1 = P(T).$$

□

Observation 3.6. *Let T be a tree that is not a star and for which $\Delta(T) \geq 3$. Then there exists $v \in V$ that has a unique non-pendant neighbor and at least one pendant neighbor.*

Proof. Let r be a vertex of degree $\Delta(T)$. Let Q be a path starting at r , and of maximum length. Denote by u_1 the terminal vertex of Q , by v the predecessor of u_1 in Q (note that $v \neq r$), and by w the predecessor of v in Q (it is possible that $w = r$). Then u_1 is a pendant neighbor of v , and w is the unique non-pendant neighbor of v . \square

Remark. A similar result appears as Lemma 13 in [S].

Proof of Theorem 3.2. As previously mentioned, the second equality comes from Observation 3.2. The proof of the first equality is by induction on $|T|$. The base of the induction is ensured by Corollary 3.3. The theorem holds for any path and any star, by Observations 3.3 and 3.4. Hence we may assume that T is not a star and $\Delta(T) \geq 3$. Let v be as in Observation 3.6, and let u_1, u_2, \dots, u_m ($m \geq 1$) be its pendant neighbors.

Case 1. $m = 1$.

In this case, $P(T - u_1) = P(T)$. By the induction hypothesis, there exists $S \subseteq V(T - u_1)$ such that $f_{T-u_1}(S) = P(T - u_1) = P(T)$. Hence,

$$f_T(S) \geq f_{T-u_1}(S) = P(T),$$

so $f_T(S) = P(T)$ by Lemma 3.5.

Case 2. $m = 2$.

In this case it is straightforward to see that $P(T - \{u_1, u_2, v\}) = P(T) - 1$. By the induction hypothesis, there exists $S \subseteq V(T - \{u_1, u_2, v\})$ such that $f_{T-\{u_1, u_2, v\}}(S) = P(T) - 1$. Hence, for $S_1 = S \cup \{v\}$ we have

$$f_T(S_1) = f_{T-\{u_1, u_2, v\}}(S) + 3 - 2 = P(T).$$

Case 3. $m \geq 3$.

In this case it is straightforward to see that $P(T - u_m) = P(T) - 1$. By

the induction hypothesis, there exists $S \subseteq V(T - u_m)$ such that $f_{T-u_m}(S) = P(T - u_m) = P(T) - 1$. If $v \in S$ then $f_T(S) = f_{T-u_m}(S) + 1 = P(T)$, so we may assume that $v \notin S$. We claim that $u_1, u_2, \dots, u_{m-1} \notin S$. Suppose otherwise that $u_1 \in S$. Then

$$f_{T-u_m}(S \setminus \{u_1\}) = f_{T-u_m}(S) + 2 - 1 > P(T - u_m),$$

contradicting Lemma 3.5. Let $S_1 = S \cup \{v\}$. Then

$$f_T(S_1) \geq f_T(S) + m - 2 \geq f_{T-u_m}(S) + 1 = P(T),$$

so $f_T(S_1) = P(T)$ by Lemma 3.5.

This completes the proof that

$$P(T) = \max_{S \subseteq V} \{f_T(S)\} = \max_{0 \leq k \leq n} \{\text{MD}_k(T) - k\}.$$

□

We pause to note a similar result to Theorem 3.2. Given a tree T , Johnson and Duarte [JD1] ascertained that $P(T)$ is the maximum of $p - q$ such that there exist q vertices whose deletion leaves p components each of which is a path (possibly including singleton paths). It is obvious that this maximum is at most $\max_{0 \leq k \leq n} \{\text{MD}_k(T) - k\}$ since any components are allowed in determining $\text{MD}_k(T)$, and the converse is also true: if any component of the remaining forest is not a path, then deleting a vertex of degree greater than 2 increases the value of $\text{MD}_k(T) - k$. The observation of Johnson and Duarte can thus be seen as a corollary of Theorem 3.2. We will see the usefulness of allowing non-path components in Section 6, where we show that the lower values of $\text{MD}_k(T)$ provide an exact description of part of the boundary of $\mathcal{I}(T)$.

Definition 3.3. Let T be a tree.

- (a) A set $S \subseteq V(T)$ is said to be *optimal* if $f_T(S) = P(T)$.

(b) Let $c(T) = \min \{|S| : S \text{ is optimal}\}$.

(c) We say S is *minimal optimal* if S is optimal and $|S| = c(T)$.

Observation 3.7. For a tree T ,

$$c(T) = \min_{0 \leq k \leq n} \{k : n - \text{MD}_k(T) + k = \text{mr}(T)\}.$$

Proof. Theorem 3.1, Theorem 3.2, Definition 3.3, and Observation 3.2. \square

Observation 3.8. For a tree T ,

$$c(T) \leq \left\lfloor \frac{\text{mr}(T)}{2} \right\rfloor.$$

Proof. Let $h = c(T)$ so $n - \text{MD}_h(T) + h = \text{mr}(T)$. Recall that $k + \text{MD}_k(T) \leq n$ for any integer k , $0 \leq k \leq n$, so in particular $h \leq n - \text{MD}_h(T)$ and therefore $2h \leq \text{mr}(T)$. \square

Observation 3.9. Let T be a tree and let $S \subseteq V(T)$ be minimal optimal. Then $d(v) \geq 3$ for every $v \in S$.

Example 3.1. We calculate $c(T)$ for paths and stars.

- $T = P_n$: Then $f_{P_n}(\emptyset) = 1 = P(P_n)$ so $c(P_n) = 0$.
- $T = S_n, n \geq 4$: Let v be the degree $n-1$ vertex of S_n . Then $f_{S_n}(\{v\}) = n - 1 - 2 + 1 = P(S_n) > 1 = f_{S_n}(\emptyset)$. So $c(S_n) = 1$.

Proposition 3.10. Let T be a tree and let $v \in V(T)$ be adjacent to $m \geq 2$ pendant vertices u_1, u_2, \dots, u_m , and at most one non-pendant vertex w . Then there is a path tree \mathcal{P} of T in which $u_1 v u_2 \in \mathcal{P}$.

Proof. The claim is obvious if T is a star, $|T| \geq 3$, so assume this is not the case. Then v is adjacent to exactly one non-pendant vertex.

Let \mathcal{P} be a path tree of T . Then at least $m-2$ of the vertices u_1, u_2, \dots, u_m give single-vertex paths in \mathcal{P} . Let Q be a path in \mathcal{P} containing v . Then Q

contains at least one pendant neighbor of v , say u_1 . Then $Q = u_1vv_1v_2 \dots v_k$. Note that $k \geq 1$, as \mathcal{P} is a path tree. If $v_1 = u_2$, then $Q = u_1vu_2$. Otherwise, u_2 is a single-vertex path in \mathcal{P} . We can form a new path tree \mathcal{P}_1 by replacing the path $u_1vv_1v_2 \dots v_k$ and the singleton path u_2 of \mathcal{P} by the pair of paths u_1vu_2 and $v_1v_2 \dots v_k$. \square

Proposition 3.11. *Let T be a tree and let $v \in V(T)$ be adjacent to $m \geq 2$ pendant vertices u_1, u_2, \dots, u_m , and at most one non-pendant vertex w . Let $T_1 = T - \{u_1, u_2, \dots, u_m, v\}$. Then*

$$P(T) = P(T_1) + m - 1,$$

and

$$c(T) \leq c(T_1) + 1.$$

If $m \geq 3$,

$$c(T) = c(T_1) + 1.$$

Proof. The proposition clearly holds if T is a star, so we may assume that this is not the case. Let \mathcal{P}_1 be a path tree for T_1 . Then $\mathcal{P}_1 \cup u_1vu_2 \cup u_3 \cup \dots \cup u_m$ is a path cover for T , so $P(T) \leq P(T_1) + m - 1$.

Now let \mathcal{P} be a path tree for T containing the path u_1vu_2 (see Proposition 3.10). Then $\mathcal{R} = \{u_1vu_2, u_3, \dots, u_m\} \subseteq \mathcal{P}$, and $\mathcal{P} \setminus \mathcal{R}$ is a path cover for T_1 . Therefore,

$$P(T_1) \leq |\mathcal{P} \setminus \mathcal{R}| = |\mathcal{P}| - (m - 1) = P(T) - (m - 1).$$

Hence $P(T) = P(T_1) + m - 1$.

Now let S be a minimal optimal set for T_1 , so $|S| = c(T_1)$. This implies that $|E_{T_1}(S)| - 2|S| + 1 = P(T_1)$. Let $S_v = S \cup \{v\}$. Since T is not a star v has a unique non-pendant neighbor w . The vertices w, u_1, u_2, \dots, u_m are adjacent to v , so $|E_T(S_v)| = |E_{T_1}(S)| + m + 1$. Then

$$\begin{aligned} f_T(S_v) &= |E_T(S_v)| - 2|S_v| + 1 = |E_{T_1}(S)| + m + 1 - 2|S| - 2 + 1 \\ &= P(T_1) + m - 1 = P(T), \end{aligned}$$

so S_v is an optimal set for T . It follows that

$$c(T) \leq |S_v| = |S| + 1 = c(T_1) + 1.$$

Now assume that $m \geq 3$ and that S is a minimal optimal set for T . By Observation 3.9, none of the vertices u_1, u_2, \dots, u_m is in S . If $v \notin S$, Lemma 3.5 implies

$$\begin{aligned} P(T) &\geq |E_T(S \cup \{v\})| - 2|S \cup \{v\}| + 1 \\ &\geq |E_T(S)| + 3 - 2|S| - 2 + 1 = P(T) + 1, \end{aligned}$$

a contradiction. Therefore $v \in S$.

Let $S' = S \setminus \{v\}$. Then $|E_{T_1}(S')| \geq |E_T(S)| - (m + 1)$, so

$$\begin{aligned} f_{T_1}(S') &\geq |E_T(S)| - (m + 1) - 2|S'| + 1 \\ &= |E_T(S)| - (m + 1) - 2|S| + 2 + 1 \\ &= P(T) - (m - 1) = P(T_1). \end{aligned}$$

It follows from Lemma 3.5 that S' is optimal for T_1 , implying

$$c(T_1) \leq |S'| = |S| - 1 = c(T) - 1.$$

Hence $c(T) = c(T_1) + 1$. □

Proposition 3.11 gives us a simple algorithm to calculate $P(T)$ and thus the minimum rank of a tree. We will use the fact that if u is a pendant vertex whose neighbor v has degree 2, then any path in a minimal path cover that includes the vertex v will also include the vertex u , and $P(T) = P(T - u)$.

Observation 3.12. *Let T be a tree. Then $P(T)$ may be calculated as follows:*

1. Set G to T and set p to 0.
2. If G has a pendant vertex u whose neighbor v has degree 2, then replace G by $G - u$ and repeat step 2.

3. If G consists of a single edge or single vertex, then $P(T) = p + 1$. If G is a star on $m + 1$ vertices, then $P(T) = p + m - 1$.
4. In all other cases (by Observation 3.6) there will be some $v \in V(G)$ that is adjacent to $m \geq 2$ pendant vertices u_1, u_2, \dots, u_m and exactly one non-pendant vertex w . Replace G by $G - \{u_1, u_2, \dots, u_m, v\}$, replace p by $p + m - 1$, and return to step 2.

The calculation of $c(T)$ is not quite as straightforward as that of $P(T)$, although we can show one special case in which it is additive on subgraphs. For this we need the following definition.

Definition 3.4. Let F and G be graphs on at least two vertices, each with a vertex labeled v . Then $F \oplus G$ is the graph on $|F| + |G| - 1$ vertices obtained by identifying the vertex v in F with the vertex v in G .

The vertex v in Definition 3.4 is commonly referred to as a *cut vertex* of the graph $F \oplus G$. The next result determines $c(T)$ when $d(v) = 2$.

Theorem 3.4. Let T_1 and T_2 be trees each with a pendant vertex labeled v . Let $T = T_1 \oplus T_2$. Then $c(T) = c(T_1) + c(T_2)$.

Proof. Let R_1, R_2 be minimal optimal sets for T_1, T_2 , respectively. Then

$$f_{T_i}(R_i) = |E_{T_i}(R_i)| - 2|R_i| + 1 = P(T_i), \quad i = 1, 2.$$

Since $P(T) \leq P(T_1) + P(T_2) - 1$, and $v \notin R_1, R_2$ by Observation 3.9, by Lemma 3.5

$$\begin{aligned} P(T) \geq f_T(R_1 \cup R_2) &= |E_T(R_1 \cup R_2)| - 2|R_1 \cup R_2| + 1 \\ &= \sum_{i=1}^2 (|E_{T_i}(R_i)| - 2|R_i| + 1) - 1 \\ &= P(T_1) + P(T_2) - 1 \geq P(T). \end{aligned}$$

Therefore, $R_1 \cup R_2$ is an optimal set for T by Lemma 3.5 and $c(T) \leq |R_1 \cup R_2| = |R_1| + |R_2| = c(T_1) + c(T_2)$. We also see that $P(T) = P(T_1) + P(T_2) - 1$.

Suppose now S is a minimal optimal set for T . By Observation 3.9, $v \notin S$. Let $S_i = S \cap V(T_i)$, $i = 1, 2$. Since $v \notin S$, $S_1 \cap S_2 = \emptyset$. Now

$$P(T) = f_T(S) = |E_T(S)| - 2|S| + 1,$$

$$P(T_1) \geq f_{T_1}(S_1) = |E_{T_1}(S_1)| - 2|S_1| + 1,$$

and

$$P(T_2) \geq f_{T_2}(S_2) = |E_{T_2}(S_2)| - 2|S_2| + 1.$$

Then

$$\begin{aligned} 1 &= P(T_1) + P(T_2) - P(T) \\ &\geq \sum_{i=1}^2 (|E_{T_i}(S_i)| - 2|S_i| + 1) - (|E_T(S)| - 2|S| + 1) = 1, \end{aligned}$$

so we must have

$$P(T_i) = f_{T_i}(S_i), \quad i = 1, 2,$$

and

$$c(T_1) + c(T_2) \leq |S_1| + |S_2| = |S| = c(T).$$

□

Corollary 3.5. *Let p be a pendant vertex in a tree T and suppose the neighbor of p has degree 2. Then $c(T) = c(T - p)$.*

Corollary 3.6. *If a tree T has exactly one vertex of degree $d > 2$, then $c(T) = 1$.*

Proof. It is straightforward to see, by repeated application of Corollary 3.5, that $c(T) = c(S_{d+1}) = 1$. □

Definition 3.5. Let T be a tree and let k be an integer such that $0 \leq k \leq c(T)$. Then

$$r_k(T) = \max \{|E_T(S)| : S \subseteq V(T), |S| = k\}.$$

Observation 3.13. For a tree T , $r_k(T) = \text{MD}_k(T) + k - 1$.

The next theorem will play an important role in simplifying the computation of $\mathcal{I}(T)$.

Theorem 3.7. Let T be a tree with $c(T) \geq 1$. Then

$$r_k(T) - r_{k-1}(T) \geq \begin{cases} 3 & \text{if } k = 1 \text{ or } k = c(T), \\ 2 & \text{if } 1 < k < c(T). \end{cases}$$

Proof. Since $c(T) \geq 1$, T is not a path. Therefore $\Delta(T) \geq 3$, implying $r_1(T) - r_0(T) = \Delta(T) - 0 \geq 3$. If $k = c(T)$,

$$r_k(T) - 2k + 1 = P(T),$$

while

$$r_{k-1}(T) - 2(k-1) + 1 < P(T).$$

Then

$$r_k(T) - r_{k-1}(T) - 2 \geq 1.$$

Thus, the stronger conclusion in the special cases $k = 1$ and $k = c(T)$ has been established. We proceed by induction on $|T|$. Since T cannot be a path, the base of the induction is $|T| = 4$, and the only relevant tree T with $|T| = 4$ is S_4 . Since $c(S_4) = 1$ the theorem holds in this case.

Consider now the general induction step. Let T be a tree on n vertices, and let $k \in \{2, \dots, c(T)-1\}$. Note that if $c(T) \leq 2$ we are done. In particular, we can assume T is not a star. We have to show that $r_k(T) - r_{k-1}(T) \geq 2$.

By Observation 3.6 there exists $v \in V$ that is adjacent to a unique non-pendant vertex w , and to pendant vertices u_1, u_2, \dots, u_m , where $m \geq 1$.

Case 1. $m \geq 2$.

Let $T_1 = T - \{u_1, u_2, \dots, u_m, v\}$. Then $c(T_1) \geq c(T) - 1$ by Proposition 3.11. This tells us both that we are allowed to assume the induction hypothesis on the tree T_1 (which requires $c(T_1) \geq 1$) and that $k \leq c(T_1)$.

Now choose $Q \subseteq V(T)$ with $|Q| \leq k - 1$ and $|E_T(Q)| \geq r_{k-1}(T)$. This choice is possible (with equality) by the definition of r_{k-1} . We can assume without loss of generality that Q contains none of the vertices $\{u_1, \dots, u_m\}$ as follows: If $v \in Q$ we delete all u_i 's that belong to Q , possibly decreasing $|Q|$ without changing $|E_T(Q)|$. If at least one of u_1, u_2, \dots, u_m , say u_1 , belongs to Q but $v \notin Q$, we replace u_1 by v in Q and delete from Q all remaining u_i , possibly decreasing $|Q|$ and possibly increasing $|E_T(Q)|$. We give the name ℓ to $|Q|$, so $\ell \leq k - 1$.

Subcase 1.1. Suppose that $v \notin Q$. Let $R = Q \cup \{v\}$. Then $|R| = \ell + 1 \leq k$, and $E_T(R) \supseteq E_T(Q) \cup \{vu_1, vu_2, \dots, vu_m\}$. Hence

$$r_k(T) \geq r_{\ell+1}(T) \geq |E_T(R)| \geq |E_T(Q)| + m \geq r_{k-1}(T) + m \geq r_{k-1}(T) + 2.$$

Subcase 1.2. Suppose that $v \in Q$. Let $Q' = Q \setminus \{v\}$. Then $|Q'| = \ell - 1 \leq k - 2$, and $r_{\ell-1}(T_1) \geq |E_{T_1}(Q')|$. By the induction hypothesis,

$$r_{k-1}(T_1) - r_{\ell-1}(T_1) \geq r_{k-1}(T_1) - r_{k-2}(T_1) \geq 2.$$

Choose $R \subseteq V(T_1)$ with $|R| = k - 1$ such that $r_{k-1}(T_1) = |E_{T_1}(R)|$. Let $R_v = R \cup \{v\}$. Then $|R_v| = k$, and

$$E_T(R_v) = E_{T_1}(R) \cup \{vu_1, vu_2, \dots, vu_m, vw\}.$$

Also,

$$E_T(Q) = E_{T_1}(Q') \cup \{vu_1, vu_2, \dots, vu_m, vw\},$$

so

$$|E_T(R_v)| = |E_{T_1}(R)| + m + 1, \quad |E_T(Q)| = |E_{T_1}(Q')| + m + 1.$$

Then

$$\begin{aligned}
r_k(T) &\geq |E_T(R_v)| = |E_{T_1}(R)| + m + 1 = r_{k-1}(T_1) + m + 1 \\
&\geq r_{\ell-1}(T_1) + 2 + m + 1 \\
&\geq |E_{T_1}(Q')| + m + 1 + 2 = |E_T(Q)| + 2 \geq r_{k-1}(T) + 2.
\end{aligned}$$

Case 2. $m = 1$.

Let $T_1 = T - u_1$. By Corollary 3.5, we have $c(T) = c(T_1)$, and clearly $P(T) = P(T_1)$. Since $k \leq c(T) - 1$, $k \leq c(T_1) - 1$.

As in Case 1, we choose $Q \subseteq V(T)$ with $|Q| \leq k - 1$ and $|E_T(Q)| \geq r_{k-1}(T)$, and can assume without loss of generality that $u_1 \notin Q$.

Subcase 2.1. Suppose that $v \notin Q$. Then $|E_{T_1}(Q)| = |E_T(Q)|$, and applying the induction hypothesis, we have

$$r_k(T) \geq r_k(T_1) \geq r_{k-1}(T_1) + 2 \geq |E_{T_1}(Q)| + 2 = |E_T(Q)| + 2 \geq r_{k-1}(T) + 2.$$

Subcase 2.2. Suppose that $v \in Q$. Let $Q' = Q \setminus \{v\}$. Then $|Q'| \leq k - 2$ and $E_{T_1}(Q') = E_T(Q')$. Hence

$$|E_{T_1}(Q')| = |E_T(Q')| \geq |E_T(Q)| - 2 \geq r_{k-1}(T) - 2.$$

Applying the induction hypothesis to T_1 ,

$$r_{k-1}(T_1) \geq r_{k-2}(T_1) + 2 \geq |E_{T_1}(Q')| + 2 \geq r_{k-1}(T).$$

Applying the induction hypothesis again to T_1 ,

$$r_k(T) \geq r_k(T_1) \geq r_{k-1}(T_1) + 2 \geq r_{k-1}(T) + 2.$$

□

Corollary 3.8. *Let T be a tree. Then for $0 \leq j \leq k \leq c(T)$,*

$$\text{MD}_k(T) \geq \text{MD}_j(T) + (k - j).$$

Proof. It suffices to prove the case $k - j = 1$. Here we have $1 \leq k \leq c(T)$, and Theorem 3.7 gives us $r_k(T) \geq r_{k-1}(T) + 2$. Making the substitution $r_k(T) = \text{MD}_k(T) + k - 1$ from Observation 3.13 gives us the desired result. \square

Proposition 3.14. *Let T be a tree on $n \geq 3$ vertices. Then $c(T) \leq \frac{n-1}{3}$.*

Proof. We prove the proposition by induction on n . The cases $n = 3, 4$ are obvious, so we consider the general induction step. The proposition holds if $T = S_n$, as $c(S_n) = 1$, so assume T is not a star. By Observation 3.6, there exists a vertex v that is adjacent to exactly one non-pendant vertex w and to pendant vertices u_1, u_2, \dots, u_m , where $m \geq 1$.

Case 1. $m = 1$.

It follows from Corollary 3.5 and the induction hypothesis that

$$c(T) = c(T - u_1) \leq \frac{n-1-1}{3} < \frac{n-1}{3}.$$

Case 2. $m \geq 2$.

Let $T_1 = T - \{u_1, u_2, \dots, u_m, v\}$. Then $|T_1| \leq n - 3$, and by Proposition 3.11 $c(T) \leq c(T_1) + 1$. Then, by induction hypothesis,

$$c(T) - 1 \leq c(T_1) \leq \frac{|T_1| - 1}{3} \leq \frac{n-4}{3} = \frac{n-1}{3} - 1.$$

\square

We conclude this section with a partial result toward the first claim of Theorem 1.1.

Definition 3.6. For a tree T we define L_T , the *minimum-rank stripe* of T , as the set

$$L_T = \{(r, s) \in \mathbb{N}_{\text{mr}(T)}^2 : r \geq c(T), s \geq c(T)\}.$$

For the moment the name “minimum-rank stripe” is not entirely justified, since it suggests that $L_T = \mathcal{I}(T) \cap \mathbb{N}_{\text{mr}(T)}^2$. In Section 6 we will show that this is the case, but we can already show one direction of containment.

Theorem 3.9. *For any tree T , $L_T \subseteq \mathcal{I}(T)$.*

Proof. Let $k = c(T)$. Given any $(r, s) \in L_T$, we have $r \geq k$, $s \geq k$, and $r + s = \text{mr}(T) = n - \text{MD}_k(T) + k$ by Observation 3.7. Then by the Stars and Stripes Lemma we have $(r, s) \in \mathcal{I}(T)$. \square

Corollary 3.10. *Theorem 1.1 gives the correct value of $\text{mr}(T)$ for T a tree.*

4 Inertia formulae for a graph with a cut vertex

In this section we interrupt our discussion of inertia sets of trees in order to derive basic formulae about the inertia set of any graph with a cut vertex. We obtain formulae for inertia sets that are the analogue of Theorem 16 in [Hs] and Theorem 2.3 in [BFH1] for minimum rank.

Definition 4.1. If Q, R are subsets of \mathbb{N}^2 , then

$$Q + R = \{(a + c, b + d) : (a, b) \in Q \text{ and } (c, d) \in R\}.$$

Addition of 3 or more sets is defined similarly.

Definition 4.2. If Q is a subset of \mathbb{N}^2 and n is a positive integer, we let

$$[Q]_n = Q \cap \mathbb{N}_{\leq n}^2.$$

We first consider the case of disconnected graphs. Since the inertia of a direct sum of matrices is the sum of the inertias of the summands, we have:

Observation 4.1. *Let $G = \bigcup_{i=1}^k G_i$. Then*

$$\mathcal{I}(G) = \mathcal{I}(G_1) + \mathcal{I}(G_2) + \cdots + \mathcal{I}(G_k),$$

and similarly for $\text{h}\mathcal{I}(G)$.

We now determine the inertia set of a graph with a cut vertex—see Definition 3.4. We first recall the following useful result [Hs], [BFH1], which reduces the minimum rank problem for graphs to the case of 2-connected graphs.

Theorem 4.1 (Hsieh; Barioli, Fallat, Hogben). *With F , G and $F \oplus_v G$ as in Definition 3.4, we have*

$$\text{mr}(F \oplus_v G) = \min\{\text{mr}(F) + \text{mr}(G), \text{mr}(F - v) + \text{mr}(G - v) + 2\}.$$

Our next result generalizes this to inertia sets.

Theorem 4.2. *Let F and G be graphs on at least two vertices with a common vertex v and let $n = |F| + |G| - 1$. Then*

$$\mathcal{I}(F \oplus_v G) = [\mathcal{I}(F) + \mathcal{I}(G)]_n \cup [\mathcal{I}(F - v) + \mathcal{I}(G - v) + \{(1, 1)\}]_n$$

and similarly for $\text{h}\mathcal{I}(F \oplus_v G)$.

Proof. We prove the complex Hermitian version of the theorem; the proof of the real symmetric version is the same but with the assumption that all matrices and vectors are real.

Let v be the last vertex of F and the first vertex of G .

Reverse containment:

I. Let $(r, s) \in [\text{h}\mathcal{I}(F) + \text{h}\mathcal{I}(G)]_n$. Then $r + s \leq n$ and there exist $(i, j) \in \text{h}\mathcal{I}(F)$ and $(k, \ell) \in \text{h}\mathcal{I}(G)$ such that $i + k = r$, $j + \ell = s$. Let

$$M = \begin{bmatrix} A & b \\ b^* & c_1 \end{bmatrix} \in \mathcal{H}(F) \quad \text{and} \quad N = \begin{bmatrix} c_2 & d^* \\ d & E \end{bmatrix} \in \mathcal{H}(G)$$

with $\text{pin}(M) = (i, j)$ and $\text{pin}(N) = (k, \ell)$, and let

$$\widehat{M} = \begin{bmatrix} A & b & 0 \\ b^* & c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \widehat{N} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_2 & d^* \\ 0 & d & E \end{bmatrix}$$

be matrices of order n . Then

$$\text{pin}(\widehat{M}) = \text{pin}(M) = (i, j), \quad \text{pin}(\widehat{N}) = \text{pin}(N) = (k, \ell),$$

and $\widehat{M} + \widehat{N} \in \mathcal{H}(F \oplus_v G)$. By the subadditivity of partial inertias (Proposition 1.5),

$$\pi(\widehat{M} + \widehat{N}) \leq \pi(\widehat{M}) + \pi(\widehat{N}) = i + k = r,$$

and

$$\nu(\widehat{M} + \widehat{N}) \leq \nu(\widehat{M}) + \nu(\widehat{N}) = j + \ell = s.$$

Since $(\pi(\widehat{M} + \widehat{N}), \nu(\widehat{M} + \widehat{N})) \in \text{h}\mathcal{I}(F \oplus_v G)$ by definition, and $r + s \leq n$, $(r, s) \in \text{h}\mathcal{I}(F \oplus_v G)$ by the Northeast Lemma (Lemma 1.1). Thus, we have $[\text{h}\mathcal{I}(F) + \text{h}\mathcal{I}(G)]_n \subseteq \text{h}\mathcal{I}(F \oplus_v G)$.

II. Now let $(r, s) \in [\text{h}\mathcal{I}(F - v) + \text{h}\mathcal{I}(G - v) + \{(1, 1)\}]_n$. Then $r + s \leq n$ and there exist $(i, j) \in \text{h}\mathcal{I}(F - v)$ and $(k, \ell) \in \text{h}\mathcal{I}(G - v)$ with $(i, j) + (k, \ell) + (1, 1) = (r, s)$. Let $A \in \mathcal{H}(F - v)$ with $\text{pin}(A) = (i, j)$ and let $E \in \mathcal{H}(G - v)$ with $\text{pin}(E) = (k, \ell)$. Choose b, c, d such that

$$M = \begin{bmatrix} A & b & 0 \\ b^* & c & d^* \\ 0 & d & E \end{bmatrix} \in \mathcal{H}(F \oplus_v G).$$

By Proposition 1.4,

$$\pi(M) \leq \pi \left(\begin{bmatrix} A & 0 \\ 0 & E \end{bmatrix} \right) + 1 = \pi(A) + \pi(E) + 1 = i + k + 1 = r,$$

and, similarly,

$$\nu(M) \leq j + \ell + 1 = s.$$

Since $(\pi(M), \nu(M)) \in \text{h}\mathcal{I}(F \oplus_v G)$, and $r + s \leq n$, by the Northeast Lemma, $(r, s) \in \text{h}\mathcal{I}(F \oplus_v G)$.

So we have

$$[\text{h}\mathcal{I}(F - v) + \text{h}\mathcal{I}(G - v) + \{(1, 1)\}]_n \subseteq \text{h}\mathcal{I}(F \oplus_v G).$$

Forward containment:

Now let $(i, j) \in \text{h}\mathcal{I}(F \oplus_v G)$. By Observation 2.1, $i + j \leq n$. Let

$$M = \begin{bmatrix} A & b & 0 \\ b^* & c & d^* \\ 0 & d & E \end{bmatrix} \in \mathcal{H}(F \oplus_v G).$$

with $\text{pin}(M) = (i, j)$. Then

$$\text{rank } A + \text{rank } E \leq \text{rank} \begin{bmatrix} A & b & 0 \\ 0 & d & E \end{bmatrix} \leq \text{rank } M \leq \text{rank } A + \text{rank } E + 2.$$

If the first and third inequalities are strict, then

$$\text{rank } A + \text{rank } E + 1 = \text{rank} \begin{bmatrix} A & b & 0 \\ 0 & d & E \end{bmatrix} = \text{rank } M.$$

The first equality implies that either $b \notin \text{Col}(A)$ or else $d \notin \text{Col}(E)$, while the second equality implies that $b \in \text{Col}(A)$ and $d \in \text{Col}(E)$. So this case does not occur and either

$$\text{rank } A + \text{rank } E = \text{rank} \begin{bmatrix} A & b & 0 \\ 0 & d & E \end{bmatrix}$$

or else

$$\text{rank } M = \text{rank } A + \text{rank } E + 2.$$

I. $\text{rank } A + \text{rank } E = \text{rank} \begin{bmatrix} A & b & 0 \\ 0 & d & E \end{bmatrix}.$

Then $b \in \text{Col}(A)$ and $d \in \text{Col}(E)$. So $b = Au$, $d = Ev$ for some $u \in \mathbb{C}^{|F|-1}$, $v \in \mathbb{C}^{|G|-1}$.

Define

$$\widehat{A} = \begin{bmatrix} A & Au \\ u^*A & u^*Au \end{bmatrix} \in \mathcal{H}(F); \quad \widehat{E} = \begin{bmatrix} v^*Ev & v^*E \\ Ev & E \end{bmatrix} \in \mathcal{H}(G).$$

Then \widehat{A} is congruent to $A \oplus \begin{bmatrix} 0 \\ 1 \times 1 \end{bmatrix}$; \widehat{E} is congruent to $E \oplus \begin{bmatrix} 0 \\ 1 \times 1 \end{bmatrix}$. Hence

$$\begin{aligned} \pi(\widehat{A}) &= \pi(A); & \nu(\widehat{A}) &= \nu(A); \\ \pi(\widehat{E}) &= \pi(E); & \nu(\widehat{E}) &= \nu(E). \end{aligned}$$

Also,

$$\text{rank } \widehat{A} = \text{rank } A \leq |F| - 1, \quad (1)$$

$$\text{rank } \widehat{E} = \text{rank } E \leq |G| - 1. \quad (2)$$

By Proposition 1.4, $\exists a, b \in \{0, 1\}$ such that

$$i = \pi(M) = \pi(A) + \pi(E) + a = \pi(\widehat{A}) + \pi(\widehat{E}) + a, \quad (3)$$

$$j = \nu(M) = \nu(A) + \nu(E) + b = \nu(\widehat{A}) + \nu(\widehat{E}) + b. \quad (4)$$

It follows from (1) and (2) that

$$\pi(\widehat{A}) + a + \nu(\widehat{A}) = \text{rank } \widehat{A} + a \leq |F| - 1 + a \leq |F|,$$

$$\pi(\widehat{E}) + \nu(\widehat{E}) + b = \text{rank } \widehat{E} + b \leq |G| - 1 + b \leq |G|.$$

Hence, by the Northeast Lemma,

$$(\pi(\widehat{A}) + a, \nu(\widehat{A})) = (\pi(A) + a, \nu(A)) \in \text{h}\mathcal{I}(F),$$

$$(\pi(\widehat{E}), \nu(\widehat{E}) + b) = (\pi(E), \nu(E) + b) \in \text{h}\mathcal{I}(G),$$

and since these two vectors add up to (i, j) , by (3) and (4), we conclude that $(i, j) \in \text{h}\mathcal{I}(F) + \text{h}\mathcal{I}(G)$.

Since $i + j \leq n$ we get $(i, j) \in [\text{h}\mathcal{I}(F) + \text{h}\mathcal{I}(G)]_n$. So in this case, $\text{h}\mathcal{I}(F \oplus_v G) \subseteq [\text{h}\mathcal{I}(F) + \text{h}\mathcal{I}(G)]_n$.

II. $\text{rank } M = \text{rank } A + \text{rank } E + 2$.

By Proposition 1.4, we have

$$\begin{aligned} i + j &\leq \pi \left(\begin{bmatrix} A & 0 \\ 0 & E \end{bmatrix} \right) + 1 + \nu \left(\begin{bmatrix} A & 0 \\ 0 & E \end{bmatrix} \right) + 1 \\ &= \pi(A) + \pi(E) + 1 + \nu(A) + \nu(E) + 1 \\ &= \pi(A) + \nu(A) + \pi(E) + \nu(E) + 2 \\ &= \text{rank}(A) + \text{rank}(E) + 2 = \text{rank } M = i + j. \end{aligned}$$

It follows that $i = \pi(A) + \pi(E) + 1$ and $j = \nu(A) + \nu(E) + 1$ and $(i, j) = (\pi(A), \nu(A)) + (\pi(E), \nu(E)) + (1, 1)$. By definition, $(i, j) \in \text{h}\mathcal{I}(F - v) + \text{h}\mathcal{I}(G -$

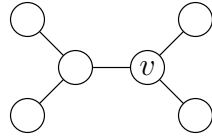
$v) + \{(1, 1)\}$, and since $i + j \leq n$, $(i, j) \in [\text{h}\mathcal{I}(F - v) + \text{h}\mathcal{I}(G - v) + \{(1, 1)\}]_n$. So in this case,

$$\text{h}\mathcal{I}(F \oplus_v G) \subseteq [\text{h}\mathcal{I}(F - v) + \text{h}\mathcal{I}(G - v) + \{(1, 1)\}]_n.$$

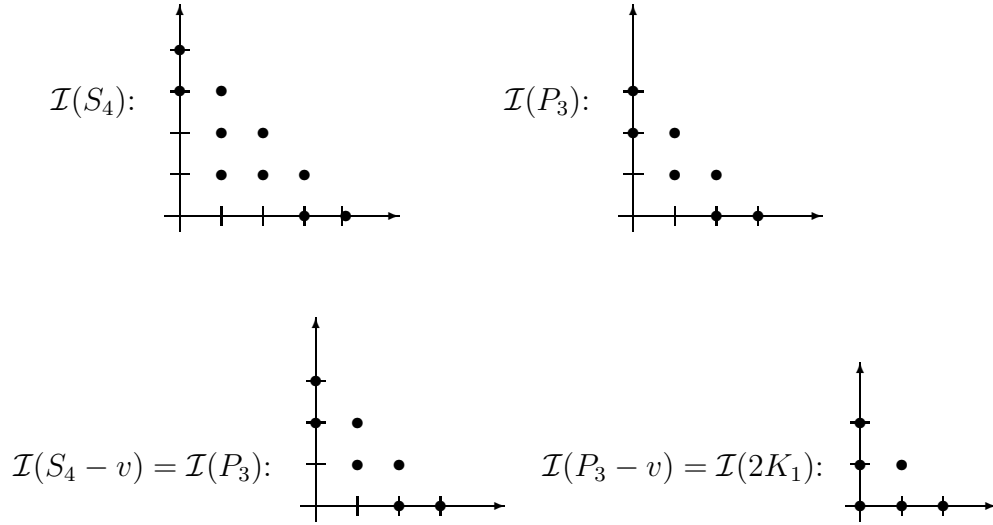
This completes the proof of the forward containment. \square

It is straightforward to show that Theorem 4.1 is a corollary of Theorem 4.2. The proof is not illuminating, so we do not include it.

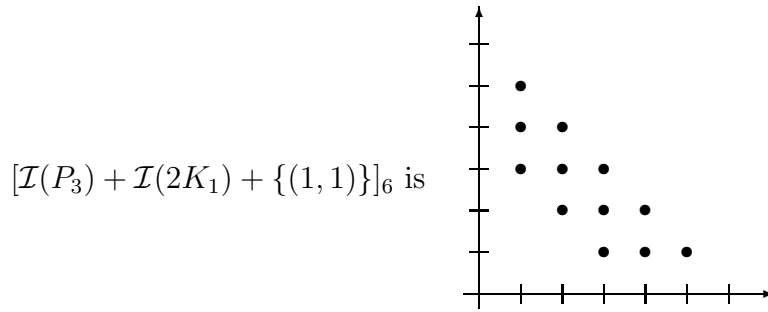
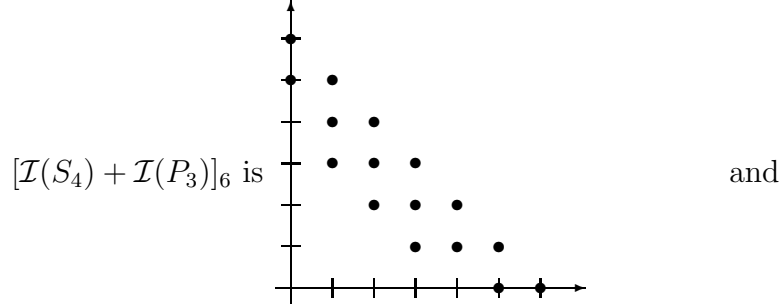
Example 4.1. Let $F = S_4$ and $G = P_3$ with v a pendant vertex in S_4 and the degree 2 vertex in P_3 . Then $T = F \oplus_v G$ is the graph below.



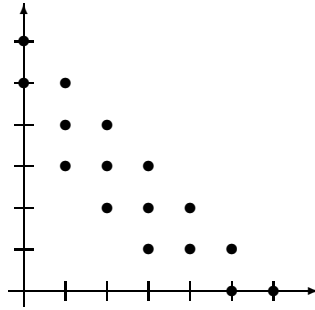
From Examples 2.4 and 2.2 we have



It follows that



Then $\mathcal{I}(T) = \mathcal{I}(F \oplus_v G) = [\mathcal{I}(S_4) + \mathcal{I}(P_3)]_6 \cup [\mathcal{I}(P_3) + \mathcal{I}(2K_1) + \{(1,1)\}]_6$ is:



Since $|T| = 6$ and $P(T) = 2$, $\text{mr}(T) = 4$ by Theorem 3.1. Since $|E_T(\{v\})| = 3$, $|E_T(\{v\})| - 2|\{v\}| + 1 = 2 = P(T)$, so $\{v\}$ is a minimal optimal set for T . We observe that in this case $L_T = \mathcal{I}(T) \cap \mathbb{N}_{\text{mr}(T)}^2$.

We pause to develop some additional fundamental properties of inertia

sets before generalizing Theorem 4.2. The next result generalizes the fact [N] that $\text{mr}(G - v) \leq \text{mr}(G) \leq \text{mr}(G - v) + 2$.

Proposition 4.2. *Let G be any graph on n vertices and let v be any vertex of G . Then we have:*

- (a) $[\mathcal{I}(G)]_{n-1} \subseteq \mathcal{I}(G - v)$.
- (b) $\mathcal{I}(G) \supseteq [\mathcal{I}(G - v)]_{n-2} + \{(1, 1)\}$.

The same inclusions hold in the Hermitian case.

Proof. Let $(r, s) \in [\mathcal{I}(G)]_{n-1}$. Then $r + s \leq n - 1$. Let $A \in \mathcal{S}(G)$ with $\text{pin}(A) = (r, s)$, and let B be the principal submatrix of A obtained by deleting the row and column v . Then $B \in \mathcal{S}(G - v)$ and by the interlacing inequalities $\text{pin}(B)$ is one of (r, s) , $(r - 1, s)$, $(r, s - 1)$, or $(r - 1, s - 1)$. Then one of these is in $\mathcal{I}(G - v)$ so by the Northeast Lemma, $(r, s) \in \mathcal{I}(G - v)$. This proves (a).

Now let $(r, s) \in [\mathcal{I}(G - v)]_{n-2}$ so $r + s \leq n - 2$. Choose $A \in \mathcal{S}(G)$ in such a way that the principal submatrix B obtained by deleting row and column v satisfies $\text{pin}(B) = (r, s)$. Then by the interlacing inequalities, $\text{pin}(A)$ is one of (r, s) , $(r + 1, s)$, $(r, s + 1)$, or $(r + 1, s + 1)$. Since $r + 1 + s + 1 \leq n$, $(r + 1, s + 1) \in \mathcal{I}(G)$ by the Northeast Lemma applied to G . This completes the proof of (b).

The proof of the Hermitian case is the same, but with Hermitian notation. □

Proposition 4.3. *If v is a pendant vertex of the graph G and $(i, j) \in \mathcal{I}(G - v)$, then $(i + 1, j) \in \mathcal{I}(G)$ and $(i, j + 1) \in \mathcal{I}(G)$, and similarly for $\text{h}\mathcal{I}(G - v)$ and $\text{h}\mathcal{I}(G)$.*

Proof. As usual, the proofs of the real symmetric and Hermitian versions do not differ materially. Let v be the first vertex of G and let its neighbor u be

the second. Let $A \in \mathcal{S}(G - v)$ with $\text{pin}(A) = (i, j)$. Then

$$M = \begin{bmatrix} J_2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \in \mathcal{S}(G)$$

and $\text{rank } M = 1 + \text{rank } A$. By Proposition 1.6

$$\pi(M) \leq \pi(A) + 1 \quad \text{and} \quad \nu(M) \leq \nu(A).$$

Then $\text{rank } M = \pi(M) + \nu(M) \leq \pi(A) + 1 + \nu(A) = \text{rank } A + 1 = \text{rank } M$ and $(i + 1, j) = (\pi(A) + 1, \nu(A)) = (\pi(M), \nu(M)) \in \mathcal{I}(G)$. Similarly, $(i, j + 1) \in \mathcal{I}(G)$. \square

The following corollary of Theorem 4.1 is very useful in simplifying the calculation of the minimum rank of a graph.

Proposition 4.4 ([S, Lemma 38]). *If the degree of v is 2 in $F \oplus_v G$, then*

$$\text{mr}(F \oplus_v G) = \text{mr}(F) + \text{mr}(G).$$

The following result generalizes this fact to inertia sets.

Proposition 4.5. *If the degree of v is 2 in $F \oplus_v G$, and $n = |F| + |G| - 1$, then*

$$\mathcal{I}(F \oplus_v G) = [\mathcal{I}(F) + \mathcal{I}(G)]_n,$$

and similarly for $\text{h}\mathcal{I}(F \oplus_v G)$.

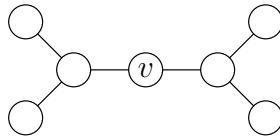
Proof. By Theorem 4.2 it suffices to show that

$$[\mathcal{I}(F - v) + \mathcal{I}(G - v) + \{(1, 1)\}]_n \subseteq [\mathcal{I}(F) + \mathcal{I}(G)]_n.$$

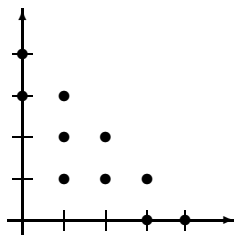
Let $(r, s) \in [\mathcal{I}(F - v) + \mathcal{I}(G - v) + \{(1, 1)\}]_n$. Then $r + s \leq n$ and $(r, s) = (i, j) + (k, \ell) + (1, 1)$ with $(i, j) \in \mathcal{I}(F - v)$ and $(k, \ell) \in \mathcal{I}(G - v)$. Since v is pendant in both F and G , by Proposition 4.3, $(i + 1, j) \in \mathcal{I}(F)$ and $(k, \ell + 1) \in \mathcal{I}(G)$, so $(r, s) = (i + 1 + k, j + \ell + 1) \in \mathcal{I}(F) + \mathcal{I}(G)$. Since $r + s \leq n$, $(r, s) \in [\mathcal{I}(F) + \mathcal{I}(G)]_n$.

Replacing \mathcal{I} by $\text{h}\mathcal{I}$ uniformly proves the Hermitian case. \square

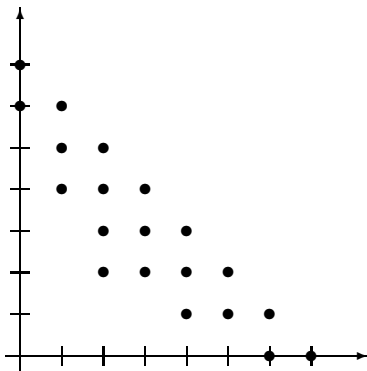
Example 4.2. Let $F = G = S_4$ and let v be a pendant vertex in each of F and G so that $T = F \oplus_v G$ is the graph below.



By Proposition 4.5, $\mathcal{I}(T) = [\mathcal{I}(F) + \mathcal{I}(G)]_7 = [\mathcal{I}(S_4) + \mathcal{I}(S_4)]_7$. Knowing that the inertia set $\mathcal{I}(S_4)$ is



allows us to calculate $\mathcal{I}(T)$, as depicted below.



Here, $\text{mr}(T) = 4$ is attained only at the partial inertia $(2, 2)$, and one can easily check that $c(T) = 2$, so that $L_T = \{(2, 2)\}$.

We close this section with the generalization of Theorem 4.2. We first extend Definition 3.4.

Definition 4.3. Let G_1, G_2, \dots, G_k , $k \geq 2$, be graphs on at least two vertices with a common vertex v and let $G = \bigoplus_{i=1}^k G_i$ be the graph on $n = \sum_{i=1}^k |G_i| - (k-1)$ vertices obtained by identifying the vertex v in each of the G_i . We call G the *vertex sum* of the graphs G_1, G_2, \dots, G_k at v .

Theorem 4.3. Let G be a graph on $n \geq 3$ vertices and let v be a cut vertex of G . Write $G = \bigoplus_{i=1}^k G_i$, $k \geq 2$, the vertex sum of G_1, G_2, \dots, G_k at v . Then

$$\begin{aligned} \mathcal{I}(G) &= [\mathcal{I}(G_1) + \mathcal{I}(G_2) + \dots + \mathcal{I}(G_k)]_n \\ &\cup [\mathcal{I}(G_1 - v) + \mathcal{I}(G_2 - v) + \dots + \mathcal{I}(G_k - v) + \{(1, 1)\}]_n, \end{aligned} \quad (5)$$

and similarly for $\text{h}\mathcal{I}(G)$.

Proof. The idea of the proof is the same as in the proof of Theorem 4.2, which is showing that each side of equation (5) is contained in the other. Since each of the theorems cited applies equally well to $\text{h}\mathcal{I}$ as to \mathcal{I} , the same proof demonstrates both cases.

Forward containment:

We prove that

$$\begin{aligned} \mathcal{I}(G) &\subseteq [\mathcal{I}(G_1) + \mathcal{I}(G_2) + \dots + \mathcal{I}(G_k)]_n \\ &\cup [\mathcal{I}(G_1 - v) + \mathcal{I}(G_2 - v) + \dots + \mathcal{I}(G_k - v) + \{(1, 1)\}]_n \end{aligned} \quad (6)$$

by induction on k . For $k = 2$ this follows from Theorem 4.2. Assume (6) holds for all integers j with $2 \leq j < k$. Let $G' = \bigoplus_{i=1}^{k-1} G_i$, the vertex sum of G_1, \dots, G_{k-1} at v and let $n' = |G'|$. Then by Theorem 4.2,

$$\begin{aligned} \mathcal{I}(G) &= \mathcal{I}(G' \oplus_v G_k) \subseteq [\mathcal{I}(G') + \mathcal{I}(G_k)]_n \\ &\cup [\mathcal{I}(G' - v) + \mathcal{I}(G_k - v) + \{(1, 1)\}]_n. \end{aligned}$$

But

$$\mathcal{I}(G' - v) = \mathcal{I}\left(\bigcup_{i=1}^{k-1} (G_i - v)\right) = \mathcal{I}(G_1 - v) + \dots + \mathcal{I}(G_{k-1} - v)$$

by Observation 4.1. Applying the induction hypothesis to $\mathcal{I}(G')$ we have

$$\begin{aligned}
\mathcal{I}(G) &\subseteq \left[\left\{ \left[\mathcal{I}(G_1) + \cdots + \mathcal{I}(G_{k-1}) \right]_{n'} \right. \right. \\
&\quad \left. \left. \cup \left[\mathcal{I}(G_1 - v) + \cdots + \mathcal{I}(G_{k-1} - v) + \{(1, 1)\} \right]_{n'} \right\} + \mathcal{I}(G_k) \right]_n \\
&\cup \left[\mathcal{I}(G_1 - v) + \cdots + \mathcal{I}(G_k - v) + \{(1, 1)\} \right]_n \\
&= \left[\left(\left[\mathcal{I}(G_1) + \cdots + \mathcal{I}(G_{k-1}) \right]_{n'} + \mathcal{I}(G_k) \right) \right. \\
&\quad \left. \cup \left(\left[\mathcal{I}(G_1 - v) + \cdots + \mathcal{I}(G_{k-1} - v) + \{(1, 1)\} \right]_{n'} + \mathcal{I}(G_k) \right) \right]_n \\
&\cup \left[\mathcal{I}(G_1 - v) + \cdots + \mathcal{I}(G_k - v) + \{(1, 1)\} \right]_n \\
&= \left[\left[\mathcal{I}(G_1) + \cdots + \mathcal{I}(G_{k-1}) \right]_{n'} + \mathcal{I}(G_k) \right]_n \\
&\cup \left[\left[\mathcal{I}(G_1 - v) + \cdots + \mathcal{I}(G_{k-1} - v) + \{(1, 1)\} \right]_{n'} + \mathcal{I}(G_k) \right]_n \\
&\cup \left[\mathcal{I}(G_1 - v) + \cdots + \mathcal{I}(G_k - v) + \{(1, 1)\} \right]_n \\
&\subseteq \left[\mathcal{I}(G_1) + \cdots + \mathcal{I}(G_k) \right]_n \\
&\cup \left[\left(\left[\mathcal{I}(G_1 - v) + \cdots + \mathcal{I}(G_{k-1} - v) \right]_{n'-2} + \{(1, 1)\} \right) + \mathcal{I}(G_k) \right]_n \\
&\cup \left[\mathcal{I}(G_1 - v) + \cdots + \mathcal{I}(G_k - v) + \{(1, 1)\} \right]_n.
\end{aligned}$$

Let

$$\begin{aligned}
Q_1 &= \left[\mathcal{I}(G_1) + \cdots + \mathcal{I}(G_k) \right]_n, \\
Q_2 &= \left[\mathcal{I}(G_1 - v) + \cdots + \mathcal{I}(G_k - v) + \{(1, 1)\} \right]_n, \\
Q_0 &= \left[\left[\mathcal{I}(G_1 - v) + \cdots + \mathcal{I}(G_{k-1} - v) \right]_{n'-2} + \{(1, 1)\} + \mathcal{I}(G_k) \right]_n.
\end{aligned}$$

We show that $Q_0 \subseteq Q_2$. Suppose that $(r, s) \in Q_0$. Then

$$(r, s) = (i_1, j_1) + (i_2, j_2) + \cdots + (i_{k-1}, j_{k-1}) + (1, 1) + (i, j)$$

with $(i_t, j_t) \in \mathcal{I}(G_t - v)$, $t = 1, \dots, k-1$, $(i, j) \in \mathcal{I}(G_k)$,

$$\sum_{t=1}^{k-1} (i_t + j_t) \leq n' - 2, \quad \text{and} \quad r + s \leq n.$$

If $i + j < |G_k|$, by Proposition 4.2(a), $(i, j) \in \mathcal{I}(G_k - v)$ and then $(r, s) \in Q_2$. So suppose that $i + j = |G_k|$. At least one of i, j is greater than 0. Without loss of generality, assume $i > 0$. By Proposition 2.3, $(i-1, j) \in \mathcal{I}(G_k)$, and by Proposition 4.2(a), $(i-1, j) \in \mathcal{I}(G_k - v)$. Since $n' = |G'| = \sum_{t=1}^{k-1} |G_t| - (k-2)$, we have

$$n' - 2 = \sum_{t=1}^{k-1} (|G_t| - 1) - 1 = \left(\sum_{t=1}^{k-1} |G_t - v| \right) - 1.$$

Therefore, $\sum_{t=1}^{k-1} (i_t + j_t) < \sum_{t=1}^{k-1} |G_t - v|$. Without loss of generality, assume $i_1 + j_1 < |G_1 - v|$. By the Northeast Lemma $(i_1 + 1, j_1) \in \mathcal{I}(G_1 - v)$. Since $r + s \leq n$, and $(r, s) = (i_1 + 1, j_1) + (i_2, j_2) + \cdots + (i_{k-1}, j_{k-1}) + (i-1, j) + (1, 1)$, we again have $(r, s) \in Q_2$. This completes the proof that $Q_0 \subseteq Q_2$. Therefore $\mathcal{I}(G) \subseteq Q_1 \cup Q_2$, which is (6).

Reverse containment:

A proof by induction is not straightforward. However, one can show the two containments $[\mathcal{I}(G_1) + \cdots + \mathcal{I}(G_k)]_n \subseteq \mathcal{I}(G)$, and $[\mathcal{I}(G_1 - v) + \cdots + \mathcal{I}(G_k - v) + \{(1, 1)\}]_n \subseteq \mathcal{I}(G)$, by simply imitating each step in the proof of Theorem 4.2. As there are no new ideas in the proof, we omit it. \square

5 The cut-vertex formula for elementary inertias

The results of the previous section give us a way to inductively calculate the inertia set of any graph once we know the inertia sets of 2-connected graphs. In this section we prove that the same inductive formula holds when calculating the set of elementary inertias. Claim 1 of Theorem 1.1 will then follow because a forest is a graph with no 2-connected subgraph on 3 or more vertices.

It is convenient to describe the elementary inertias of a graph G in terms of bicolored edge-colorings of certain subgraphs of G .

Definition 5.1. Let G be a graph on n vertices, let S be a subset of $V(G)$, and let X and Y be disjoint subsets of $E(G-S)$. The ordered triple (S, X, Y) is called a *bicolored span of G* if $(V \setminus S, X \cup Y)$ is a spanning forest of $G - S$. (A *spanning forest* of a graph consists of a spanning tree for each connected component.) If (S, X, Y) is a bicolored span of G , we say that the ordered pair $(|S| + |X|, |S| + |Y|)$ is a *color vector* of G . The set of color vectors of G is denoted $\mathcal{C}(G)$.

The color vector counts how many edges of the spanning forest have been marked with either the first color or the second color, but it also counts the set S of excluded vertices twice, as though each such vertex were marked simultaneously with both colors. Because every spanning forest has the same number of edges, the quantity $|X| + |Y|$ depends only on S , and for a given size $|S| = k$, $|X| + |Y|$ is minimized if $G - S$ has $\text{MD}_k(G)$ components. If G is a graph on n vertices, $\ell = \text{MD}_0(G)$ is the number of components of G , and (\emptyset, X, Y) is a bicolored span of G , then $|X| + |Y| + \ell = n$.

Observation 5.1. If G is a graph on n vertices and $\ell = \text{MD}_0(G)$, then $\mathbb{N}_{n-\ell}^2 \subseteq \mathcal{C}(G)$.

Definition 5.2. If Q is a subset of \mathbb{N}^2 , we define the *northeast expansion* of Q as

$$Q^{\nearrow} = Q + \mathbb{N}^2.$$

For example, the Northeast Lemma is equivalent to the statement that, for G a graph on n vertices, $[\mathcal{I}(G)^{\nearrow}]_n \subseteq \mathcal{I}(G)$. The prevalence of northeast expansions in this section leads us to define the following equivalence relation:

Definition 5.3. Given two sets $P, Q \subseteq \mathbb{N}^2$, we say that P is *northeast equivalent* to Q , written as $P \sim Q$, if $P^{\nearrow} = Q^{\nearrow}$.

Definition 5.4. Let G be a graph on n vertices, and let (x, y) be an ordered pair of integers. We say that (x, y) is a *northeast color vector* of G if $x + y \leq n$ and if $x \geq x_0$ and $y \geq y_0$ for some color vector (x_0, y_0) of G .

Note that the set of all northeast color vectors of G is $[\mathcal{C}(G)^{\nearrow}]_n$. The term *northeast color vector* is actually a synonym for *elementary inertia*, as we now demonstrate.

Proposition 5.2. *Let G be a graph on n vertices. Then $\mathcal{E}(G) = [\mathcal{C}(G)^{\nearrow}]_n$.*

Proof. We show both inclusions.

Forward inclusion. Let (r, s) be an elementary inertia of G . Then there exist a nonnegative integer k and an ordered pair of integers (r_0, s_0) such that

$$k \leq r_0 \leq r, \quad k \leq s_0 \leq s, \quad \text{and} \quad r_0 + s_0 = n - \text{MD}_k(G) + k.$$

Let S be chosen such that $|S| = k$ and $G - S$ has $\text{MD}_k(G)$ components, and let F be a spanning forest of $G - S$, so that F has $n - k$ vertices and $r_0 + s_0 - 2k$ edges. We partition the edges of F into two sets X and Y with $r_0 - k$ and $s_0 - k$ edges respectively. It follows that $(k + |X|, k + |Y|) = (r_0, s_0)$ is a color vector of G . Since $r + s \leq n$, (r, s) belongs to the set $[\mathcal{C}(G)^{\nearrow}]_n$ of northeast color vectors of G .

Reverse inclusion. Let (x, y) be a northeast color vector of G , and let (S, X, Y) be a bicolored span of G such that $x_0 = |S| + |X| \leq x$ and $y_0 = |S| + |Y| \leq y$. Letting $k = |S|$, we can assume without loss of generality that S is chosen among all sets of size k in such a way as to minimize $|X| + |Y|$, or in other words that $G - S$ has $\text{MD}_k(G)$ components. Under this assumption we have $|X| + |Y| + \text{MD}_k(G) = n - k$, so $n - \text{MD}_k(G) + k = x_0 + y_0 \leq x + y \leq n$. We further have $k \leq x_0 \leq x$ and $k \leq y_0 \leq y$, so (x, y) is an elementary inertia of G . \square

We now state some set-theoretic results that allow us to simplify certain expressions involving Q^{\nearrow} and $[Q]_n$.

Observation 5.3. *For $Q \subseteq \mathbb{N}^2$ and nonnegative integers $m \leq n$, we have*

$$1. \quad \left[[Q]_n \right]_m = \left[[Q]_m \right]_n = [Q]_m.$$

2. $[[Q^\nearrow]_n]^\nearrow_m = [Q^\nearrow]_m$.
3. $[Q^\nearrow]_m \sim [Q]_m$.
4. If P is a stripe of rank m , then $[Q + P]_n = [Q]_{n-m} + P$.
5. $\mathbb{N}_m^2 \subseteq Q$ implies $Q \sim [Q]_m$.
6. $Q \sim [Q]_m$ implies $Q \sim [Q]_n$.

Proof. These are all straightforward consequences of the definitions. \square

Proposition 5.4. *Let ℓ , m , and n be nonnegative integers with $0 \leq \ell \leq n$ and $0 \leq m \leq n$, suppose that $Q \subseteq \mathbb{N}^2$ satisfies $Q \sim [Q]_{n-\ell}$, and let $P = [Q^\nearrow]_n$. Then*

1. $P \sim [P]_{n-\ell}$,
2. $[P^\nearrow]_m = [P]_m$, and
3. $[P^\nearrow]_n = P$.

Proof. We have

$$P = [Q^\nearrow]_n \sim [Q]_n \sim Q \sim [Q]_{n-\ell} \sim [Q^\nearrow]_{n-\ell} = [[Q^\nearrow]_n]_{n-\ell} = [P]_{n-\ell},$$

$$[P^\nearrow]_m = [[Q^\nearrow]_n]^\nearrow_m = [Q^\nearrow]_m = [[Q^\nearrow]_n]_m = [P]_m,$$

and

$$[P^\nearrow]_n = [[Q^\nearrow]_n]^\nearrow_n = [Q^\nearrow]_n = P.$$

\square

We can apply this proposition immediately. First note that Observations 5.1 and 5.3 (5) give us

Observation 5.5. *Let G be a graph on n vertices with ℓ components. Then $\mathcal{C}(G) \sim [\mathcal{C}(G)]_{n-\ell}$.*

Observation 5.5 and Proposition 5.2 allow us to apply Proposition 5.4, by substituting $\mathcal{C}(G)$ for Q .

Observation 5.6 (Northeast equivalence I). *Let G be a graph on n vertices with ℓ components, and let m be an integer in the range $0 \leq m \leq n$. Then*

1. $\mathcal{E}(G) \sim [\mathcal{E}(G)]_{n-\ell}$,
2. $[\mathcal{E}(G)^\nearrow]_m = [\mathcal{E}(G)]_m$, and
3. $[\mathcal{E}(G)^\nearrow]_n = \mathcal{E}(G)$.

Observation 5.6 (3) can be viewed as a Northeast Lemma for elementary inertias.

Lemma 5.7. *Let Q_1, \dots, Q_k be subsets of \mathbb{N}^2 , and suppose that for some collection n_1, \dots, n_k of nonnegative integers we have $Q_i \sim [Q_i]_{n_i}$ for $i = 1, \dots, k$. Let $Q = Q_1 + \dots + Q_k$ and let $n = n_1 + \dots + n_k$. Then*

$$[Q^\nearrow]_n = [Q_1^\nearrow + \dots + Q_k^\nearrow]_n = [Q_1^\nearrow]_{n_1} + \dots + [Q_k^\nearrow]_{n_k}.$$

Proof. The first equality comes from the observation that $\mathbb{N}^2 + \mathbb{N}^2 = \mathbb{N}^2$.

For the second equality, the reverse inclusion is easy to check. Suppose then that we are given

$$(x, y) \in [Q_1^\nearrow + Q_2^\nearrow + \dots + Q_k^\nearrow]_n,$$

so there exist k ordered pairs of integers $(x_i, y_i) \in Q_i^\nearrow$ with $x = \sum_{i=1}^k x_i$, $y = \sum_{i=1}^k y_i$, and $x + y \leq n$. For any such collection $\{(x_i, y_i)\}$, we can define two quantities, a *surplus*

$$s = \sum_{i=1}^m \max(x_i + y_i - n_i, 0)$$

and a *deficit*

$$d = \sum_{i=1}^m \max(n_i - x_i - y_i, 0),$$

so that $x + y - s + d = n$ and hence $s \leq d$. If $s = 0$, then in every case we have $(x_i, y_i) \in [Q_i^\nearrow]_{n_i}$, so

$$(x, y) \in [Q_1^\nearrow]_{n_1} + [Q_2^\nearrow]_{n_2} + \cdots + [Q_k^\nearrow]_{n_k}$$

and we are done. But we can assume $s = 0$ without loss of generality for the following reason: If $s > 0$, then $d > 0$ also and for some integers i and j in the range $1 \leq i, j \leq k$ we have $x_i + y_i > n_i$ and $x_j + y_j < n_j$. Since $Q_i^\nearrow = [Q_i]_{n_i}^\nearrow$, we can replace (x_i, y_i) by either $(x_i - 1, y_i)$ or $(x_i, y_i - 1)$, one of which must belong to Q_i^\nearrow , and simultaneously replace (x_j, y_j) with respectively either $(x_j + 1, y_j) \in Q_j^\nearrow$ or $(x_j, y_j + 1) \in Q_j^\nearrow$. This reduces both the value of s and the value of d , so we can assume without loss of generality that $s = 0$, giving the desired result. \square

The following proposition is an immediate corollary.

Proposition 5.8. *Given $Q \subseteq \mathbb{N}^2$ and nonnegative integers $m \leq n$, suppose that $Q \sim [Q]_m$. Then*

$$[Q^\nearrow]_n = [Q^\nearrow]_m + \sum_{i=1}^{n-m} \mathcal{E}(K_1).$$

Proof. Apply Lemma 5.7 with $k = n - m + 1$, $Q_1 = Q$, $n_1 = m$, and for $i > 1$, $Q_i = \{(0, 0)\}$ and $n_i = 1$. We have abbreviated $[\{(0, 0)\}^\nearrow]_1$ by the equivalent expression $\mathcal{E}(K_1)$. \square

With the necessary set-theoretic tools in place, we can proceed to demonstrate some properties of $\mathcal{E}(G)$, starting with the fact that it is additive on the connected components of G .

Proposition 5.9 (Additivity on components). *Let $G = \bigcup_{i=1}^k G_i$. Then*

$$\mathcal{E}(G) = \mathcal{E}(G_1) + \mathcal{E}(G_2) + \cdots + \mathcal{E}(G_k).$$

Proof. We first observe that for any bicolored span (S, X, Y) of G , each entry of the triple is a disjoint union of corresponding entries from bicolored spans of the components G_i , so

$$\mathcal{C}(G) = \mathcal{C}(G_1) + \mathcal{C}(G_2) + \cdots + \mathcal{C}(G_k).$$

Now let $n = |G|$ and for each integer i in the range $1 \leq i \leq k$, let $n_i = |G_i|$. From Observations 5.5 and 5.3 (6) we can conclude that $\mathcal{C}(G_i) \sim [\mathcal{C}(G_i)]_{n_i}$. Since $n = n_1 + n_2 + \cdots + n_k$, we can apply Lemma 5.7 to obtain

$$[\mathcal{C}(G)^\nearrow]_n = [\mathcal{C}(G_1)^\nearrow]_{n_1} + [\mathcal{C}(G_2)^\nearrow]_{n_2} + \cdots + [\mathcal{C}(G_k)^\nearrow]_{n_k},$$

which by Proposition 5.2 is equivalent to the desired conclusion. \square

Before stating and proving the cut vertex formula for elementary inertia sets, it will be useful to split the set $\mathcal{E}(G)$ into two specialized sets depending on a choice of vertex v , and establish some of the properties of these sets.

Definition 5.5. Let G be a graph and let v be a vertex of G .

- If (S, X, Y) is a bicolored span of G and $v \in S$, then we say that the ordered pair $(|S| + |X|, |S| + |Y|)$ is a *v-deleting color vector* of G . The set of *v-deleting color vectors* of G is denoted $\mathcal{C}_v^-(G)$.
- If (S, X, Y) is a bicolored span of G and $v \notin S$, then we say that the ordered pair $(|S| + |X|, |S| + |Y|)$ is a *v-keeping color vector* of G . The set of *v-keeping color vectors* of G is denoted $\mathcal{C}_v^+(G)$.

Definition 5.6. Let G be a graph on n vertices including v . We define the set of *v-deleting elementary inertias* of G as

$$\mathcal{E}_v^-(G) = [\mathcal{C}_v^-(G)^\nearrow]_n$$

and the set of *v-keeping elementary inertias* of G as

$$\mathcal{E}_v^+(G) = [\mathcal{C}_v^+(G)^\nearrow]_n.$$

The first result we need is an immediate consequence of these definitions.

Proposition 5.10 (Splitting at v). *Let G be a graph with $v \in V(G)$. Then*

$$\mathcal{E}(G) = \mathcal{E}_v^-(G) \cup \mathcal{E}_v^+(G).$$

There are equivalent, simpler expressions for the set of v -deleting color vectors and v -deleting elementary inertias of G .

Proposition 5.11 (The v -deleting formula). *Let G be a graph on $n \geq 2$ vertices with $v \in V(G)$. Then*

$$\mathcal{C}_v^-(G) = \mathcal{C}(G - v) + \{(1, 1)\}$$

and

$$\mathcal{E}_v^-(G) = [\mathcal{E}(G - v)]_{n-2} + \{(1, 1)\} = [\mathcal{E}(G - v) + \{(1, 1)\}]_n.$$

Proof. The triple (S, X, Y) is a bicolored span of G with $v \in S$ if and only if the triple $(S - \{v\}, X, Y)$ is a bicolored span of $G - v$. It follows that the v -deleting color vectors (r, s) in $\mathcal{C}_v^-(G)$ are exactly the vectors $(1 + x, 1 + y)$ where (x, y) is a color vector of $G - v$. This gives us our first conclusion

$$\mathcal{C}_v^-(G) = \mathcal{C}(G - v) + \{(1, 1)\}.$$

With the first conclusion as our starting point, we now have

$$\mathcal{E}_v^-(G) = [\mathcal{C}_v^-(G)]_n^{\nearrow} = [\mathcal{C}(G - v)]_n^{\nearrow} + \{(1, 1)\}_n.$$

Since $\{(1, 1)\}$ is a stripe of rank 2, by Observation 5.3 this simplifies to

$$\begin{aligned} \mathcal{E}_v^-(G) &= [\mathcal{C}(G - v)]_{n-2}^{\nearrow} + \{(1, 1)\} \\ &= \left[[\mathcal{C}(G - v)]_{n-1}^{\nearrow} \right]_{n-2} + \{(1, 1)\} \\ &= [\mathcal{E}(G - v)]_{n-2} + \{(1, 1)\} \\ &= [\mathcal{E}(G - v) + \{(1, 1)\}]_n \end{aligned}$$

which completes the proof. □

Observation 5.12. *Let G be a graph whose n vertices include v , and let ℓ be the number of components of $G - v$. Then $\mathcal{C}_v^-(G) \sim [\mathcal{C}_v^-(G)]_{n+1-\ell}$.*

Proof. By Observation 5.5, $\mathcal{C}(G - v) \sim [\mathcal{C}(G - v)]_{n-1-\ell}$. Proposition 5.11 and Observation 5.3 (4) then give us $\mathcal{C}_v^-(G) \sim [\mathcal{C}_v^-(G)]_{n+1-\ell}$. \square

Substituting $Q = \mathcal{C}_v^-(G)$ into Proposition 5.4 now gives us a result about v -deleting elementary inertias.

Observation 5.13 (Northeast equivalence II). *Let G be a graph whose n vertices include v , let ℓ be the number of components of $G - v$, and let m be an integer in the range $0 \leq m \leq n$. Then*

1. $\mathcal{E}_v^-(G) \sim [\mathcal{E}_v^-(G)]_{n+1-\ell}$,
2. $[\mathcal{E}_v^-(G)]_m^\nearrow = [\mathcal{E}_v^-(G)]_m$, and
3. $[\mathcal{E}_v^-(G)]_n^\nearrow = \mathcal{E}_v^-(G)$.

Similar results hold for the v -keeping color vectors and v -keeping elementary inertias:

Observation 5.14. *Let G be a graph whose n vertices include v , and let $\ell = \text{MD}_0(G)$. Then $\mathcal{C}_v^+(G) \sim [\mathcal{C}_v^+(G)]_{n-\ell}$.*

Proof. It suffices to consider bicolored spans of the form (\emptyset, X, Y) , which of course satisfy $v \notin \emptyset$. The set of v -keeping color vectors arising from such bicolored spans is exactly $\mathbb{N}_{n-\ell}^2$, from which the desired result follows by Observation 5.3 (5). \square

Proposition 5.4 now gives us:

Observation 5.15 (Northeast equivalence III). *Let G be a graph whose n vertices include v , let $\ell = \text{MD}_0(G)$, and let m be an integer in the range $0 \leq m \leq n$. Then*

1. $\mathcal{E}_v^+(G) \sim [\mathcal{E}_v^+(G)]_{n-\ell}$,

2. $[\mathcal{E}_v^+(G)^\nearrow]_m = [\mathcal{E}_v^+(G)]_m$, and
3. $[\mathcal{E}_v^+(G)^\nearrow]_n = \mathcal{E}_v^+(G)$.

It is possible to restrict the set of allowable bicolored spans that define $\mathcal{C}_v^+(G)$ and still obtain the full set of v -keeping color vectors of G .

Proposition 5.16. *Let G be a graph with vertex v , and let $E' = E(G - v)$. Suppose that (x, y) belongs to $\mathcal{C}_v^+(G)$. Then there exists a bicolored span (S, X, Y) of G with $v \notin S$ such that $(x, y) = (|S| + |X|, |S| + |Y|)$ and such that $(S, X \cap E', Y \cap E')$ is a bicolored span of $G - v$.*

Proof. By the definition of $\mathcal{C}_v^+(G)$, there exists a bicolored span (S, X, Y) of G with $v \notin S$ such that $(x, y) = (|S| + |X|, |S| + |Y|)$. The vertex v thus belongs to some component G_i of $G - S$, and those edges in X and Y which are part of G_i give a spanning tree T_i of G_i . There is no loss of generality if we assume that T_i is constructed as follows: First, a spanning tree is obtained for each component of $G_i - v$. Each subtree is then connected to v by way of a single edge, so that the degree of v in T_i is equal to $\text{MD}_0(G_i - v)$. With this assumption, $(S, X \cap E', Y \cap E')$ is a bicolored span of $G - v$. \square

The next key ingredient is a consequence of Propositions 5.11 and 5.16.

Proposition 5.17 (Domination by $G - v$). *Let G be a graph on n vertices, one of which is v . Then for $\epsilon \in \{-, +\}$ we have*

$$[\mathcal{E}_v^\epsilon(G)]_{n-1} \subseteq \mathcal{E}(G - v).$$

Given Proposition 5.10, Proposition 5.17 is equivalent to an inclusion on elementary inertia sets which has already been proven for inertia sets as Proposition 4.2 (a):

Proposition 5.18. *For any graph G and any vertex $v \in V(G)$,*

$$[\mathcal{E}(G)]_{n-1} \subseteq \mathcal{E}(G - v).$$

We need one more result before stating and proving the cut vertex formula for elementary inertias.

Proposition 5.19 (The v -keeping cut vertex formula). *Let $G = \bigoplus_{i=1}^k G_i$ be a graph on n vertices which is a vertex sum of graphs G_1, G_2, \dots, G_k at v , for $k \geq 2$. Then*

$$\mathcal{E}_v^+(G) = [\mathcal{E}_v^+(G_1) + \mathcal{E}_v^+(G_2) + \dots + \mathcal{E}_v^+(G_k)]_n.$$

Proof. Let G , v , n , and G_1, \dots, G_k be as in the statement of the proposition. We first establish a related identity,

$$\mathcal{C}_v^+(G) = \mathcal{C}_v^+(G_1) + \mathcal{C}_v^+(G_2) + \dots + \mathcal{C}_v^+(G_k).$$

This holds because

1. The sets $V(G_i) - \{v\}$ are disjoint, and their union is $V(G) - \{v\}$, so subsets $S \subseteq V(G)$ with $v \notin S$ are in bijective correspondence with collections of subsets $S_i \subseteq V(G_i)$ none of which contain v .
2. For any such set S partitioned as a union of S_i , $G - S$ is a vertex sum at v of the graphs $G_i - S_i$, and so the set $E(G - S)$ is a disjoint union of $E(G_i - S_i)$.
3. A subgraph F of the vertex sum $G - S$ is a spanning forest of $G - S$ if and only if F is a vertex sum of graphs F_i each of which is a spanning forest of $G_i - S_i$.

For each graph G_i , let $n_i = |G_i|$, so that $(n - 1) = \sum_{i=1}^k (n_i - 1)$. Since each graph G_i contains the vertex v , $\text{MD}_0(G_i) \geq 1$. Observations 5.14 and 5.3 (6) then give us $\mathcal{C}_v^+(G_i) \sim [\mathcal{C}_v^+(G_i)]_{n_i-1}$. Thus by Lemma 5.7 we have

$$[\mathcal{C}_v^+(G)]_{n-1}^{\nearrow} = [\mathcal{C}_v^+(G_1)]_{n_1-1}^{\nearrow} + \dots + [\mathcal{C}_v^+(G_k)]_{n_k-1}^{\nearrow}.$$

We also have $\mathcal{C}_v^+(G) \sim [\mathcal{C}_v^+(G)]_{n-1}$, so by Proposition 5.8 we can add k copies of $\mathcal{E}(K_1)$ to both sides to obtain

$$[\mathcal{C}_v^+(G)]_{n-1+k}^\nearrow = [\mathcal{C}_v^+(G_1)]_{n_1}^\nearrow + \cdots + [\mathcal{C}_v^+(G_k)]_{n_k}^\nearrow$$

which gives the desired formula by Observation 5.3 (1) and Definition 5.6. \square

The proof of the cut vertex formula depends on the following properties of $\mathcal{E}(G)$, $\mathcal{E}_v^+(G)$, and $\mathcal{E}_v^-(G)$:

- Northeast equivalence I and III (Observations 5.6 and 5.15),
- Additivity on components (Proposition 5.9),
- Splitting at v (Proposition 5.10),
- The v -deleting formula (Proposition 5.11),
- Domination by $G - v$ (Proposition 5.17), and
- The v -keeping cut vertex formula (Proposition 5.19).

Theorem 5.1 (The cut vertex formula for elementary inertias). *Let G be a graph on $n \geq 3$ vertices and let v be a cut vertex of G . Write $G = \bigoplus_{i=1}^k G_i$, $k \geq 2$, the vertex sum of G_1, G_2, \dots, G_k at v . Then*

$$\begin{aligned} \mathcal{E}(G) &= [\mathcal{E}(G_1) + \mathcal{E}(G_2) + \cdots + \mathcal{E}(G_k)]_n \\ &\cup [\mathcal{E}(G_1 - v) + \mathcal{E}(G_2 - v) + \cdots + \mathcal{E}(G_k - v) + \{(1, 1)\}]_n. \end{aligned}$$

Proof. We manipulate both sides to obtain the same set.

Define two sets

$$Q^- = [\mathcal{E}(G_1 - v) + \cdots + \mathcal{E}(G_k - v) + \{(1, 1)\}]_n$$

and

$$Q^+ = [\mathcal{E}_v^+(G_1) + \cdots + \mathcal{E}_v^+(G_k)]_n.$$

By Propositions 5.11 and 5.9, $\mathcal{E}_v^-(G) = Q^-$ and by Proposition 5.19, $\mathcal{E}_v^+(G) = Q^+$, so by Proposition 5.10, $\mathcal{E}(G) = Q^- \cup Q^+$.

The right hand side is

$$\text{RHS} = [\mathcal{E}(G_1) + \cdots + \mathcal{E}(G_k)]_n \cup Q^-.$$

For each $i = 1, \dots, k$, let $n_i = |G_i|$, so that $\mathcal{E}(G_i) \sim [\mathcal{E}(G_i)]_{n_i-1}$ (Observations 5.6 (1) and 5.3 (6), since in each case $\ell \geq 1$). Starting with Observation 5.6 (3) and then applying Lemma 5.7 both backwards and forwards, we have

$$\begin{aligned} [\mathcal{E}(G_1) + \cdots + \mathcal{E}(G_k)]_n &= \left[[\mathcal{E}(G_1)]^{\nearrow}_{n_1} + \cdots + [\mathcal{E}(G_k)]^{\nearrow}_{n_k} \right]_n \\ &= \left[[\mathcal{E}(G_1)]^{\nearrow} + \cdots + [\mathcal{E}(G_k)]^{\nearrow} \right]_{n-1+k} \\ &= [\{(0,0)\}^{\nearrow} + \mathcal{E}(G_1)^{\nearrow} + \cdots + \mathcal{E}(G_k)^{\nearrow}]_n \\ &= \mathcal{E}(K_1) + [\mathcal{E}(G_1)]^{\nearrow}_{n_1-1} + \cdots + [\mathcal{E}(G_k)]^{\nearrow}_{n_k-1} \end{aligned}$$

which by Observation 5.6 (2) gives us

$$\text{RHS} = Q^- \cup \left(\mathcal{E}(K_1) + [\mathcal{E}(G_1)]_{n_1-1} + \cdots + [\mathcal{E}(G_k)]_{n_k-1} \right).$$

By applying Proposition 5.10 to each term $[\mathcal{E}(G_i)]_{n_i-1}$ we obtain

$$\text{RHS} = Q^- \cup \left(\mathcal{E}(K_1) + \sum_{i=1}^k ([\mathcal{E}_v^-(G_i)]_{n_i-1} \cup [\mathcal{E}_v^+(G_i)]_{n_i-1}) \right).$$

For any $\alpha = (\epsilon_1, \dots, \epsilon_m) \in \{-, +\}^k$ we will define

$$\mathcal{E}_v^\alpha = \sum_{i=1}^k [\mathcal{E}_v^{\epsilon_i}(G_i)]_{n_i-1}.$$

This gives us

$$\text{RHS} = \bigcup_{\alpha \in \{-, +\}^k} Q^- \cup (\mathcal{E}(K_1) + \mathcal{E}_v^\alpha).$$

We divide the 2^k choices for α into two cases: either ϵ_j is “−” for some $j \in \{1, \dots, k\}$, or ϵ_i is “+” for all i . In the first case, by Proposition 5.11 we have

$$\begin{aligned} \mathcal{E}(K_1) + \mathcal{E}_v^\alpha &= \mathcal{E}(K_1) + [\mathcal{E}_v^{\epsilon_1}(G_1)]_{n_1-1} + \dots \\ &\quad \dots + [\mathcal{E}(G_j - v) + \{(1, 1)\}]_{n_j-1} + \dots \\ &\quad \dots + [\mathcal{E}_v^{\epsilon_k}(G_k)]_{n_k-1}. \end{aligned}$$

We wish to show that this is a subset of Q^- . For every i besides j , we have $[\mathcal{E}_v^{\epsilon_i}(G_i)]_{n_i-1} \subseteq \mathcal{E}(G_i - v)$ by Proposition 5.17. The remaining terms we regroup as

$$\begin{aligned} \mathcal{E}(K_1) + [\mathcal{E}(G_j - v) + \{(1, 1)\}]_{n_j-1} &= \mathcal{E}(K_1) + [\mathcal{E}(G_j - v)]_{n_j-3} + \{(1, 1)\} \\ &\subseteq \mathcal{E}(K_1) + [\mathcal{E}(G_j - v)]_{n_j-2} + \{(1, 1)\}. \end{aligned}$$

Observation 5.6 and Proposition 5.8 give us

$$\mathcal{E}(K_1) + [\mathcal{E}(G_j - v)]_{n_j-2} = \mathcal{E}(G_j - v).$$

We have thus shown that

$$\mathcal{E}(K_1) + \mathcal{E}_v^\alpha \subseteq \mathcal{E}(G_1 - v) + \dots + \mathcal{E}(G_k - v) + \{(1, 1)\},$$

and since

$$\mathcal{E}(K_1) + \mathcal{E}_v^\alpha = [\mathcal{E}(K_1) + \mathcal{E}_v^\alpha]_n,$$

this gives us $Q^- \cup (\mathcal{E}(K_1) + \mathcal{E}_v^\alpha) = Q^-$ in the case where α has at least one sign $\epsilon_j = \text{“−”}$.

This leaves the case where α has all signs $\epsilon_j = \text{“+”}$. By Observations 5.15 (1) and 5.3 (6), $\mathcal{E}_v^+(G_i) \sim [\mathcal{E}_v^+(G_i)]_{n_i-1}$. Starting with Observation 5.15 (2), applying Lemma 5.7 both backwards and forwards, and finally

using Observation 5.15 (3), we have

$$\begin{aligned}
\mathcal{E}(K_1) + \mathcal{E}_v^\alpha &= \mathcal{E}(K_1) + [\mathcal{E}_v^+(G_1)^\nearrow]_{n_1-1} + \cdots + [\mathcal{E}_v^+(G_k)^\nearrow]_{n_k-1} \\
&= [\{(0,0)\}^\nearrow + \mathcal{E}_v^+(G_1)^\nearrow + \cdots + \mathcal{E}_v^+(G_k)^\nearrow]_n \\
&= \left[[\mathcal{E}_v^+(G_1)^\nearrow + \cdots + \mathcal{E}_v^+(G_k)^\nearrow]_{n-1+k} \right]_n \\
&= \left[[\mathcal{E}_v^+(G_1)^\nearrow]_{n_1} + \cdots + [\mathcal{E}_v^+(G_k)^\nearrow]_{n_k} \right]_n \\
&= Q^+.
\end{aligned}$$

The entire union thus collapses to $\text{RHS} = Q^- \cup Q^+$, and the left and right hand expressions are equal. \square

Remark. We can generalize the splitting of $\mathcal{E}(G)$ into $\mathcal{E}_v^+(G)$ and $\mathcal{E}_v^-(G)$ for non-elementary inertias: Given a graph G with vertex v and $A \in \mathcal{H}(G)$, order the vertices of G such that $v = 1$ and decompose A as

$$A = \begin{bmatrix} a_{11} & b^* \\ b & B \end{bmatrix}.$$

If b is in the column space of B , then say that $\text{pin}(A) \in \text{h}\mathcal{I}_v^+(G)$, and define $\text{h}\mathcal{I}_v^-(G)$ as $[\text{h}\mathcal{I}(G - v) + \{(1,1)\}]_n$. Define $\mathcal{I}_v^+(G)$ and $\mathcal{I}_v^-(G)$ analogously. Under these definitions we can uniformly replace \mathcal{E} with $\text{h}\mathcal{I}$ or \mathcal{I} in Observations 5.6, and 5.15 and in each of Propositions 5.9, 5.10, 5.11, 5.17, and 5.19, and we claim that in every case the result still holds. We will not prove these statements, as we already have a proof of Theorem 4.3, but given those observations and propositions, the proof of Theorem 5.1 demonstrates the same cut vertex formula for inertia sets.

We now state and prove the main result of the section.

Theorem 5.2. *For any tree T , $\mathcal{I}(T) = \mathcal{E}(T)$.*

Proof. Let $n = |T|$.

If $n = 1$, $T = K_1$ and $\mathcal{I}(T) = \mathbb{N}_{[0,1]}^2$. Since $(\emptyset, \emptyset, \emptyset)$ is a bicolored span of K_1 ,

the origin $(0, 0)$ is a color vector of T and $\mathcal{E}(T) = \mathbb{N}_{[0,1]}^2$ also. If $n = 2$, then $T = K_2$, and $\mathcal{I}(K_2) = \mathbb{N}_{[1,2]}^2 = \mathcal{E}(K_2)$.

Proceeding by induction, assume that $\mathcal{I}(T) = \mathcal{E}(T)$ for all trees T on fewer than n vertices and let T be a tree on n vertices, $n \geq 3$. Let v be a cut vertex of T of degree $k \geq 2$. Write $T = \bigoplus_{i=1}^k T_i$, the vertex sum of T_1, \dots, T_k at v . By Theorem 4.3,

$$\begin{aligned} \mathcal{I}(T) &= [\mathcal{I}(T_1) + \dots + \mathcal{I}(T_k)]_n \\ &\cup [\mathcal{I}(T_1 - v) + \dots + \mathcal{I}(T_k - v) + \{(1, 1)\}]_n \end{aligned}$$

and by Theorem 5.1,

$$\begin{aligned} \mathcal{E}(T) &= [\mathcal{E}(T_1) + \dots + \mathcal{E}(T_k)]_n \\ &\cup [\mathcal{E}(T_1 - v) + \dots + \mathcal{E}(T_k - v) + \{(1, 1)\}]_n. \end{aligned}$$

Corresponding terms on the right hand side of these last two equations are equal by the induction hypothesis, so $\mathcal{I}(T) = \mathcal{E}(T)$. \square

Corollary 5.3. *For any forest F , $\mathcal{I}(F) = \mathcal{E}(F)$.*

Proof. By Theorem 5.2, $\mathcal{I}(T) = \mathcal{E}(T)$ for every component T of F , and by additivity on components for both $\mathcal{I}(G)$ (Observation 4.1) and $\mathcal{E}(G)$ (Proposition 5.9), $\mathcal{I}(F) = \mathcal{E}(F)$. \square

Claim 1 of Theorem 1.1, which says $\mathcal{I}(F) \subseteq \mathcal{E}(F)$ for any forest F , has now been verified.

We restate Theorem 1.1 compactly as

Theorem 5.4. *Let G be a graph. Then $\mathcal{I}(G) = \mathcal{E}(G)$ if and only if G is a forest.*

6 Graphical determination of the inertia set of a tree

Tabulating the full inertia set of a tree T on n vertices by means of Theorem 1.1 appears, potentially, to require a lot of calculation: Every integer k in the range $0 \leq k \leq n$ with $\text{MD}_k(T) \geq k$ gives a trapezoid (possibly degenerate) of elementary inertias, and the full elementary inertia set is the union of those trapezoids. (One could also construct every possible bicolored span of the tree, which is even more cumbersome.) In fact the calculation is quite straightforward once we have the first few values of $\text{MD}_k(T)$. In this section we present the necessary simplifications and perform the calculation for a few examples.

Definition 6.1. For any graph G on n vertices and $k \in \{0, \dots, n\}$, let

$$\pi_k(G) = \min \{r : (r, k) \in \mathcal{I}(G)\},$$

$$\nu_k(G) = \min \{s : (k, s) \in \mathcal{I}(G)\}.$$

Since $\pi_k(G) = \nu_k(G)$ for each k , we will deal exclusively with $\pi_k(G)$.

The main simplification toward calculating the inertia set of a tree is the following:

Theorem 6.1. *Let T be a tree on n vertices and let $k \in \{0, 1, \dots, c(T)\}$. Then*

$$\pi_k(T) = n - \text{MD}_k(T). \tag{7}$$

Proof. For k in the given range, we can apply Corollary 3.8 with $j = 0$ to obtain $\text{MD}_k(T) \geq \text{MD}_0(T) + k$ and in particular $\text{MD}_k(T) \geq k$. We can thus apply the Stars and Stripes Lemma to obtain $(n - \text{MD}_k(T), k) \in \mathcal{I}(T)$ and hence $\pi_k(T) \leq n - \text{MD}_k(T)$. It remains to prove, for $k \leq c(T)$,

$$n - \text{MD}_k(T) \leq \pi_k(T).$$

Suppose by way of contradiction that $(r, s) \in \mathcal{I}(T)$ with $s = k$ and $r < n - \text{MD}_k(T)$. By Theorem 1.1, every element of $\mathcal{I}(T)$ is an elementary inertia, and thus there is some integer j for which $j \leq r$, $j \leq s$, and $n - \text{MD}_j(T) + j \leq r + s$. This implies, for $0 \leq j \leq k \leq c(T)$, that

$$\text{MD}_k(T) < \text{MD}_j(T) + (k - j),$$

which contradicts Corollary 3.8. \square

Corollary 6.2. $\left\{ \pi_k(T) \right\}_{k=0}^{\text{mr}(T)-c(T)}$ is a strictly decreasing sequence.

Proof. This follows from Theorem 6.1, Corollary 3.8 (with $k - j = 1$), and Theorem 3.9. \square

Theorem 6.3. Let T be a tree. Then $L_T = \mathcal{I}(T) \cap \mathbb{N}_{\text{mr}(T)}^2$.

In other words, every partial inertia of minimum rank is in the minimum-rank stripe already defined.

Proof. We already have $L_T \subseteq \mathcal{I}(T)$ by Theorem 3.9, and $L_T \subseteq \mathbb{N}_{\text{mr}(T)}^2$ by definition. To show equality, it suffices by symmetry (Observation 2.7) to show that for $k < c(T)$, $k + \pi_k(T) > \text{mr}(T)$. Let $c = c(T)$. If $c = 0$, we are done. By Observation 3.7, $n - \text{MD}_c(T) + c = \text{mr}(T)$ but $n - \text{MD}_k(T) + k > \text{mr}(T)$ for $k < c$. It follows by Theorem 6.1 that $k + \pi_k(T) > \text{mr}(T)$ for $k < c$, which completes the proof. \square

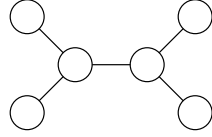
Theorem 4.2 already gives a method for determining the inertia set $\mathcal{I}(T)$ for any tree T , but with Theorem 1.1 and the simplifications above there is a much easier method, which we summarize in the following steps:

1. Use the algorithm of Observation 3.12 to find $P(T)$.
2. Since T is connected, $\text{MD}_0(T) = 1$. If T is a path then $c(T) = 0$; otherwise $c(T) \geq 1$ and $\text{MD}_1(T) = \Delta(T)$. Continue to calculate higher values of $\text{MD}_k(T)$ until $\text{MD}_k(T) - k = P(T)$, at which point $k = c(T)$.

3. The defining southwest corners of $\mathcal{I}(T)$ are $(n - \text{MD}_k(T), k)$ and its reflection $(k, n - \text{MD}_k(T))$, for $0 \leq k < c(T)$, together with the stripe L_T of partial inertias from $(n - P(T) - c(T), c(T))$ to $(c(T), n - P(T) - c(T))$.
4. Every other point of $\mathcal{I}(T)$ is a result of the Northeast Lemma applied to the defining southwest corners.

We give three examples.

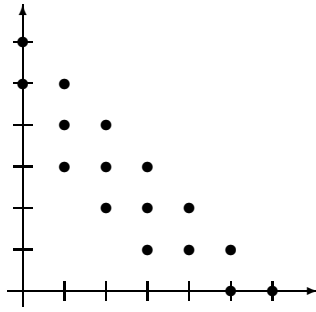
Example 6.1. Let T be the tree in Example 4.1, whose inertia set we have already calculated.



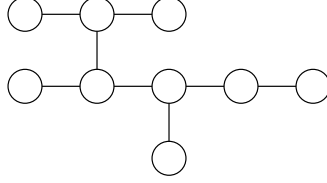
Here $P(T) = 2$ and $\text{mr}(T) = 4$. We have

$$\text{MD}_1(T) = 3, \text{MD}_1(T) - 1 = 2,$$

so $c(T) = 1$, and from $\pi_0(T) = 5$ we go immediately to L_T , starting at height 1, which is the convex stripe of three partial inertias from $(3, 1)$ to $(1, 3)$.



Example 6.2. Let T be the tree



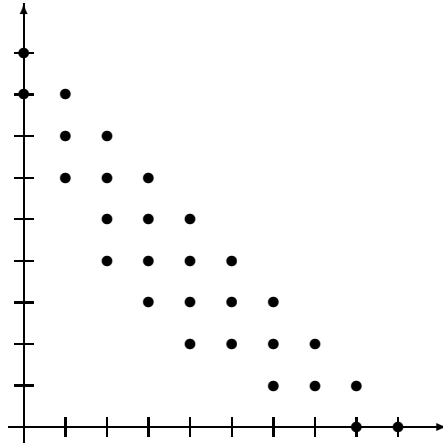
whose horizontal paths realize the path cover number $P(T) = 3$, so $\text{mr}(T) = 6$. Taking any vertex of degree 3 we have

$$\text{MD}_1(T) = 3, \text{MD}_1(T) - 1 = 2,$$

and taking the non-adjacent pair of degree-3 vertices we have

$$\text{MD}_2(T) = 5, \text{MD}_2(T) - 2 = 3,$$

so $c(T) = 2$. Starting as always from $\pi_0(T) = n - 1 = 8$, we need only one more value $\pi_1(T) = 9 - 3 = 6$ before reaching the minimum-rank stripe L_T from $(4, 2)$ to $(2, 4)$. The complete set $\mathcal{I}(T)$ is:



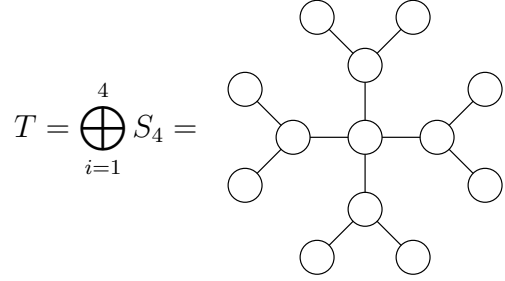
The examples we have shown so far appear to exhibit some sort of convexity. For F a forest we do at least have convexity of $\mathcal{I}(F)$ on stripes of

fixed rank, as stated in Corollary 1.2. Based on small examples one may be led to believe that, in addition, $\pi_k(T)$ is a convex function in the range $0 \leq k \leq c(T)$, or in other words that

$$\pi_k(T) - \pi_{k+1}(T) \leq \pi_{k-1}(T) - \pi_k(T) \quad \text{for } 0 < k < c(T).$$

However, this is not always the case, as seen in the following example:

Example 6.3. Given S_4 with v a pendant vertex, let T be the tree constructed as a vertex sum of four copies of the marked S_4 :



Here $P(T) = 5$ and $\text{mr}(T) = 8$. To find $\text{MD}_1(T)$ we always take a vertex of maximum degree; here

$$\text{MD}_1(T) = 4, \quad \text{MD}_1(T) - 1 = 3.$$

For $\text{MD}_2(T)$ we can either add the center of a branch or leave out the degree-4 vertex and take two centers of branches; either choice gives us

$$\text{MD}_2(T) = 5, \quad \text{MD}_2(T) - 2 = 3.$$

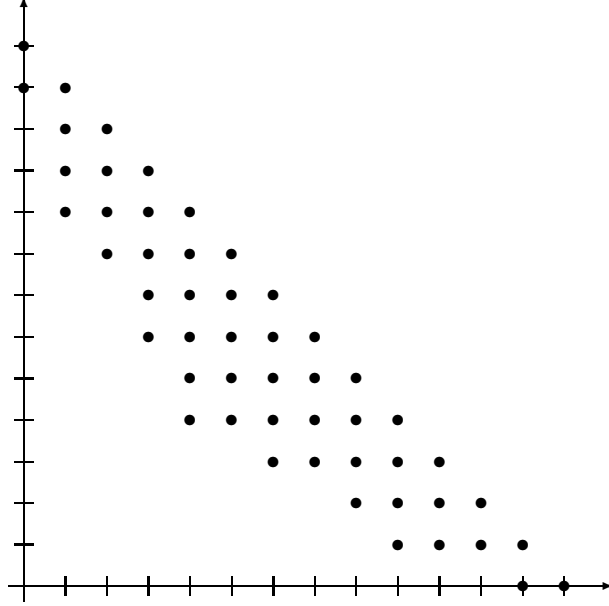
At $k = 3$ something odd happens: to remove 3 vertices and maximize the number of remaining components, we must not include the single vertex of maximum degree. Taking the centers of three branches, we obtain

$$\text{MD}_3(T) = 7, \quad \text{MD}_3(T) - 3 = 4,$$

and finally taking all four vertices of degree 3 gives us

$$\text{MD}_4(T) = 9, \quad \text{MD}_4(T) - 4 = 5 = P(T),$$

so $c(T) = 4$. The sequence $\pi_k(T)$ thus starts $(12, 9, 8, 6, 4)$. As is the case with the stars S_n and Example 4.2, we here have a tree where the minimum-rank stripe L_T is a singleton, in this case the point $(4, 4)$. The full plot of $\mathcal{I}(T)$ is:



While $\pi_k(T)$ is not a convex function over the range $0 \leq k \leq c(T)$ in the last example, the calculated set $\mathcal{I}(T)$ does at least contain all of the lattice points in its own convex hull. To expect this convexity to hold for every tree would be overly optimistic, however: if we carry out the same calculation for the larger tree $\bigoplus_{i=1}^5 S_4$ (on 16 vertices instead of 13) we find that the points $(11, 1)$ and $(5, 5)$ both belong to the inertia set, but their midpoint $(8, 3)$ does not.

Question 3. What is the computational complexity of determining the partial inertia set of a tree? The examples above pose no difficulty, but they do show that the greedy algorithm for $\text{MD}_k(T)$ fails even for T a tree. Computing all n values of $\text{MD}_k(G)$ for a general graph G is NP-hard because it can

be used to calculate the independence number: $k + \text{MD}_k(G) = n$ if and only if there is an independent set of size $n - k$.


In the next section we will consider more general graphs, rather than restricting to trees and forests, and we will see that even convexity of partial inertias within a single stripe can fail in the broader setting.

7 Beyond the forest


In this section we investigate, over the set of all graphs, what partial inertia sets—or more specifically, what complements of partial inertia sets—can occur. Once a graph G is allowed to have cycles, we can no longer assume that $\text{h}\mathcal{I}(G) = \mathcal{I}(G)$ by diagonal congruence. It happens, however, that each of the results in this section is the same in the complex Hermitian case as in the real symmetric case. For the two versions of each question we will therefore demonstrate whichever is the more difficult of the two, proving theorems over the complex numbers but providing counterexamples over the reals.

It is convenient at this point to introduce a way of representing the complements of partial inertia sets.

Definition 7.1. A *partition* is a finite (weakly) decreasing sequence of positive integers. The first integer in the sequence is called the *width* of the partition, and the number of terms in the sequence is called the *height* of the partition.

It is traditional to depict partitions with box diagrams. In order to agree with our diagrams of partial inertia sets, we choose the convention of putting the longest row of boxes on the bottom of the stack; for example, the decreasing sequence $(5, 4, 1)$ is shown as the partition . Given a box diagram of height h and width w , we index the rows by $0, 1, \dots, h - 1$ from bottom to top and the columns by $0, 1, \dots, w - 1$ from left to right.

Definition 7.2. Given a partition $\pi = (\pi_0, \pi_1, \dots, \pi_{k-1})$, let $\ell = \pi_0$, and for $i \in \{0, 1, \dots, \ell - 1\}$ let $\pi_i^* = |\{j : \pi_j \geq i + 1\}|$, i.e. the number of boxes in column i of the box diagram of π . Then $\pi^* = (\pi_0^*, \pi_1^*, \dots, \pi_{\ell-1}^*)$ is called the conjugate partition of π . A partition π is *symmetric* if $\pi = \pi^*$.

For example, we have $(5, 4, 1)^* = (3, 2, 2, 2, 1)$ and $(3, 3, 2)^* = (3, 3, 2)$, so the partition with box diagram  is symmetric. It is easy to recognize symmetric partitions visually, since a partition is symmetric if and only if its box diagram has a diagonal axis of symmetry.

In this section we will describe $\mathcal{I}(G)$, for a graph G on n vertices, in terms of its complement $\mathbb{N}_{\leq n}^2 \setminus \mathcal{I}(G)$. Definition 6.1 gives us a natural way to describe the shape of this complement as a partition. We first extend to the Hermitian case (distinguishing from the real symmetric case as usual by prepending an ‘h’).

Definition 7.3. For any graph G on n vertices and $k \in \{0, \dots, n\}$, let

$$\mathbf{h}\pi_k(G) = \min \{i : (i, k) \in \mathbf{h}\mathcal{I}(G)\}.$$

The partition corresponding to a partial inertia set is a list of as many of the values of $\pi_i(G)$ as are positive.

Definition 7.4. Given a graph G , let $k = \pi_0(G)$ and let $h = \mathbf{h}\pi_0(G)$. Then the *inertial partition* of G , denoted $\pi(G)$, is the partition

$$(\pi_0(G), \pi_1(G), \dots, \pi_{k-1}(G)).$$

The *Hermitian inertial partition* of G , denoted $\mathbf{h}\pi(G)$, is the partition

$$(\mathbf{h}\pi_0(G), \mathbf{h}\pi_1(G), \dots, \mathbf{h}\pi_{h-1}(G)).$$

It would perhaps be more accurate to call these the *partial inertia complement partition* and *Hermitian partial inertia complement partition*, but we opt for the abbreviated names.

The Northeast Lemma ensures that the inertial partition and Hermitian inertial partition of a graph are in fact partitions, and by Observation 2.7 the partitions $\pi(G)$ and $\text{h}\pi(G)$ are always symmetric. This symmetry is the reason why $k = \pi_0(G)$ is the correct point of truncation: $\pi_{k-1}(G) > 0$, but $\pi_k(G) = 0$.

Remark. If one starts with the entire first quadrant of the plane \mathbb{R}^2 and then removes everything “northeast” of any point belonging to $\mathcal{I}(G)$, the remaining “southwest complement” has the same shape as the box diagram of $\pi(G)$. The same applies of course to $\text{h}\mathcal{I}(G)$ and $\text{h}\pi(G)$.

The partial inertia sets $\mathcal{I}(G)$ and $\text{h}\mathcal{I}(G)$ can be reconstructed from the partitions $\pi(G)$ and $\text{h}\pi(G)$, respectively, if the number of vertices of G is also known. The addition of an isolated vertex to a graph G does not change $\pi(G)$.

We begin to investigate the following problem:

Question 4 (Inertial Partition Classification Problem). For which symmetric partitions π does there exist a graph G for which $\pi(G) = \pi$?

Rather than examining all possible partial inertias for a particular graph, we are now examining what restrictions on partial inertias (or rather excluded partial inertias) may hold over the class of all graphs.

The Hermitian Inertial Partition Classification Problem is the same question with $\text{h}\pi(G)$ in the place of $\pi(G)$. While it is known that there are graphs G for which $\mathcal{I}(G)$ is a strict subset of $\text{h}\mathcal{I}(G)$, it is not known whether there are partitions π that are inertial partitions but not Hermitian inertial partitions, or vice versa.

At the moment we are only able to give a complete answer to the Inertial Partition Classification Problem for symmetric partitions of height no greater than 3. We first list examples for a few symmetric partitions that are easily obtained. Of course, adding an isolated vertex to any example gives another

example for the same partition. For simplicity we will identify the partition $\pi(G)$ with its box diagram.

- For height 0, $\pi(G)$ is the empty partition if G has no edges.
- For height 1, $\pi(K_n) = \square$ for any $n > 1$.
- For height 2, $\pi(P_3) = \begin{smallmatrix} \square & \\ & \square \end{smallmatrix}$.
- For height 3, $\pi(S_4) = \begin{smallmatrix} \square & & \\ & \square & \\ & & \square \end{smallmatrix}$ and $\pi(P_4) = \begin{smallmatrix} \square & & \\ & \square & \\ & & \square \end{smallmatrix}$.

The partitions already listed cover every possible case, up to height 3, of an inertia-balanced graph, and leave three non-inertia-balanced partitions unaccounted for:

$$\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}, \text{ and } \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{smallmatrix}.$$

The following theorem eliminates cases $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$ and $\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{smallmatrix}$, as well as every larger square partition.

Theorem 7.1. *Let G be a graph and let $M \in \mathcal{H}(G)$ be a Hermitian matrix with partial inertia $(k, 0)$, $k > 1$. Then there exists a matrix $M' \in \mathcal{H}(G)$ with partial inertia (r, s) satisfying $r < k$ and $s < k$. Furthermore, if M is real then M' can be taken to be real.*

Corollary 7.2 (No Square Partitions). *For any $k > 1$, the square partition $\pi = (k, k, \dots, k)$ of height k and width k is not the inertial partition of any graph G , and is not the Hermitian inertial partition of any graph G .*

Proof of Theorem 7.1. Let G be a graph on n vertices and suppose that $M \in \mathcal{H}(G)$ is a Hermitian matrix with partial inertia $(k, 0)$. The matrix $M = [m_{ij}]$ is thus positive semidefinite of rank k , and can be factored as A^*A for some $k \times n$ complex matrix $A = [a_{ij}]$. If M is real symmetric, then A can be taken to be real.

We wish to construct a matrix $M' \in \mathcal{H}(G)$ with strictly fewer than k positive eigenvalues and also strictly fewer than k negative eigenvalues. By

Proposition 1.5, we will have accomplished our purpose if we can find $(k - 1) \times n$ matrices $B = [b_{ij}]$ and $C = [c_{ij}]$ such that $B^*B - C^*C \in \mathcal{H}(G)$, with the requirement that B and C be real if M is real.

We need to impose some mild general-position requirements on the first two rows of the matrix A , which we accomplish by replacing A by UA , where U is a unitary matrix and where U is real (and hence orthogonal) in the case that A is real. This is a permissible substitution because $(UA)^*UA = A^*U^*UA = A^*A = M$.

The first general-position requirement is that, for integers $1 \leq j \leq n$, $a_{1j} \neq 0$ and $a_{2j} \neq 0$ unless column j of A is the zero column. The second requirement, which we will justify more carefully, is that the set of ratios $\{a_{1j}/a_{2j}\}$ be disjoint from the set of conjugate reciprocals $\{\bar{a}_{2i}/\bar{a}_{1i}\}$, or equivalently

$$\bar{a}_{1i}a_{1j} \neq \bar{a}_{2i}a_{2j}$$

for any $1 \leq i, j \leq n$ where neither i nor j corresponds to a zero column.

Now we prove the existence of a unitary matrix U with the desired properties. To do so, we temporarily reserve the symbol $i \in \mathbb{C}$ to represent a solution to $i^2 + 1 = 0$. For the duration of this argument, j will represent any integer $1 \leq j \leq n$ such that column j of A is not the zero column.

Let x_j represent the vector $\begin{bmatrix} a_{1j} \\ a_{2j} \end{bmatrix}$. If \mathbb{C}^* represents the set of nonzero complex numbers, then our first general position assumption already guarantees $x_j \in (\mathbb{C}^*)^2$. We now define three functions $z, \bar{z}, w : (\mathbb{C}^*)^2 \rightarrow \mathbb{C}^*$ by

$$z\left(\begin{bmatrix} p \\ q \end{bmatrix}\right) = p/q, \quad \bar{z}\left(\begin{bmatrix} p \\ q \end{bmatrix}\right) = \bar{p}/\bar{q}, \quad \text{and} \quad w\left(\begin{bmatrix} p \\ q \end{bmatrix}\right) = \bar{q}/\bar{p}.$$

Our task is to find a unitary matrix U_1 , orthogonal in the case that A is real, such that the sets $\{z(U_1x_j)\}$ and $\{w(U_1x_j)\}$ are disjoint. In this case we can achieve the desired general position of A by replacing it with UA , where $U = U_1 \oplus I_{k-2}$.

Now consider the unitary matrices

$$R_\theta = \begin{bmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{bmatrix} \quad \text{and} \quad Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}.$$

These matrices transform complex ratios as follows:

$$z(R_\theta x) = e^{i\theta} z(x), \quad z(Qx) = \frac{z(x) + i}{iz(x) + 1}.$$

We have $Qx_j \in (\mathbb{C}^*)^2$ as long as $z(x_j) \notin \{i, -i\}$ and in particular as long as $z(x_j)$ is not pure imaginary, which is automatically true in the case A is real. In the case where A is not real, we can assume without loss of generality that no $z(x_j)$ is pure imaginary after uniformly multiplying on the left by an appropriate choice of R_θ .

Given x and y in $(\mathbb{C}^*)^2$ such that neither $z(x)$ nor $z(y)$ is pure imaginary, $z(x) = w(y)$ if and only if $z(Qx) = \bar{z}(Qy)$. We have reduced the problem to that of finding a unitary matrix U_1 , orthogonal in the case A is real, such that the sets $\{z(QU_1x_j)\}$ and $\{\bar{z}(QU_1x_j)\}$ are disjoint. In fact we will establish the stronger condition that the two finite subsets of the unit circle

$$\left\{ \frac{z(QU_1x_j)}{|z(QU_1x_j)|} \right\} \quad \text{and} \quad \left\{ \frac{\bar{z}(QU_1x_j)}{|\bar{z}(QU_1x_j)|} \right\}$$

are disjoint. Let

$$U_1 = Q^* R_\theta Q = \begin{bmatrix} \cos \theta/2 & -\sin \theta/2 \\ \sin \theta/2 & \cos \theta/2 \end{bmatrix}.$$

Then U_1 is orthogonal, and

$$\frac{z(QU_1x_j)}{|z(QU_1x_j)|} = e^{i\theta} \frac{z(Qx_j)}{|z(Qx_j)|}.$$

Our general-position requirement for A thus reduces to the following fact: Given a finite subset P of the unit circle, there is some θ such that $e^{i\theta}P$ is disjoint from its set of conjugates $e^{-i\theta}\overline{P}$ and from the set $\{i, -i\}$. To be concrete, if ϵ is the minimum nonzero angle between any element of P and

any element of \overline{P} or $\{i, -i\}$, $\theta = \epsilon/3$ will suffice. This concludes the argument justifying our assumption of general position for A .

We now construct the matrices B and C . Each column j of the matrices B and C for $1 \leq j \leq n$ is as follows:

- $b_{1j} = a_{1j}^2$.
- $b_{ij} = a_{1j}a_{(i+1)j}$ for $i \in \{2, \dots, k-1\}$.
- $c_{1j} = a_{2j}^2$.
- $c_{ij} = a_{2j}a_{(i+1)j}$ for $i \in \{2, \dots, k-1\}$.

Now consider an arbitrary entry m'_{ij} of the matrix $M' = B^*B - C^*C$; this takes the form

$$\begin{aligned} m'_{ij} &= \overline{a}_{1i}^2 a_{1j}^2 + \overline{a}_{1i} a_{1j} \overline{a}_{3i} a_{3j} + \dots + \overline{a}_{1i} a_{1j} \overline{a}_{ki} a_{kj} \\ &\quad - \overline{a}_{2i}^2 a_{2j}^2 - \overline{a}_{2i} a_{2j} \overline{a}_{3i} a_{3j} - \dots - \overline{a}_{2i} a_{2j} \overline{a}_{ki} a_{kj}, \end{aligned}$$

which factors as

$$\begin{aligned} m'_{ij} &= (\overline{a}_{1i} a_{1j} - \overline{a}_{2i} a_{2j})(\overline{a}_{1i} a_{1j} + \overline{a}_{2i} a_{2j} + \overline{a}_{3i} a_{3j} + \dots + \overline{a}_{ki} a_{kj}) \\ &= (\overline{a}_{1i} a_{1j} - \overline{a}_{2i} a_{2j}) m_{ij}. \end{aligned}$$

In case either column i or column j of A is the zero column, we have $m'_{ij} = 0 = m_{ij}$, and in all other cases we have, by the generic requirement

$$\overline{a}_{1i} a_{1j} \neq \overline{a}_{2i} a_{2j},$$

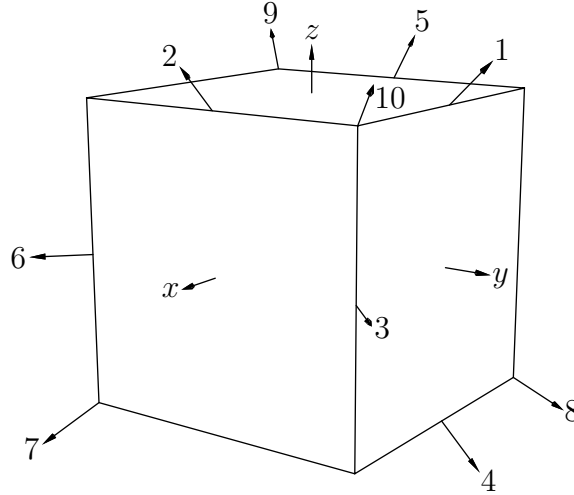
that $m'_{ij} = 0$ if and only if $m_{ij} = 0$. It follows that M' is a matrix in $\mathcal{H}(G)$, and by construction M' has at most $k-1$ positive eigenvalues and at most $k-1$ negative eigenvalues. Furthermore, if M is real symmetric then so is M' . \square

We have determined which inertial partitions occur for all partitions up to height 3 except for one: the partition $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$. Perhaps surprisingly, there is indeed a graph, on 12 vertices, that achieves this non-inertia-balanced partition in both the real symmetric and Hermitian cases.

Theorem 7.3. *There exists a graph G_{12} on 12 vertices such that $\pi(G_{12}) = \text{h}\pi(G_{12}) = \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$, the partition $(3, 3, 2)$.*

The counterexample graph G_{12} will be defined directly in terms of a real symmetric matrix with partial inertia $(3, 0)$; we will then show that $(2, 1)$ is not in $\text{h}\mathcal{I}(G_{12})$.

Consider a cube centered at the origin of \mathbb{R}^3 , and choose a representative vector for each line that passes through an opposite pair of faces, edges, or corners of the cube.



These 13 vectors give us the columns of a matrix

$$M_{13} = \begin{bmatrix} x & y & z & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 1 & 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & -1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & -1 & 1 & 0 & -1 & -1 & 1 & 1 \end{bmatrix},$$

which columns we index by the set of symbols $\{x, y, z, 1, \dots, 10\}$. The matrix $M_{13}^T M_{13}$ is real symmetric and positive semidefinite of rank 3, and thus has partial inertia $(3, 0)$. We define G_{13} as the graph on 13 vertices (labeled by the same 13 symbols) for which $M_{13}^T M_{13} \in \mathcal{S}(G_{13})$; distinct vertices i and j of G_{13} are adjacent if and only if columns i and j of M_{13} are not orthogonal. We note in passing that the subgraph of G_{13} induced by vertices labeled 1–10 is the line graph of K_5 , or the complement of the Petersen Graph. We now define the graph G_{12} (as promised in Theorem 7.3) as the induced subgraph of G_{13} obtained by deleting the vertex labeled 10.

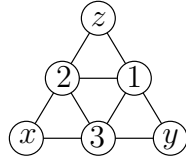
Before proving Theorem 7.3, we prove a lemma about a smaller graph G_{10} that is obtained from G_{12} by deleting the vertices labeled 6 and 9 (while retaining the labels of the other vertices). The vertices of G_{10} are thus labeled $\{x, y, z, 1, 2, 3, 4, 5, 7, 8\}$ (notice that this set skips index 6).

Lemma 7.1. *Let $A = [a_{ij}]$ be a Hermitian matrix in $\mathcal{H}(G_{10})$ of rank no more than 3. Then the first two diagonal entries a_{xx} and a_{yy} are both nonzero and have the same sign.*

Proof. Let d_x , d_y , and d_z be the first three diagonal entries of A :

$$d_x = a_{xx}, \quad d_y = a_{yy}, \quad d_z = a_{zz}.$$

We show first that all three of these entries are nonzero. For this purpose it suffices to consider only the first six rows and columns of A , corresponding to the graph $G_6 =$



sometimes called the supertriangle graph. The automorphism group of G_6 realizes any permutation of the vertices $\{x, y, z\}$ (as do the automorphism groups of G_{13} and of G_{12} , but not that of G_{10}).

The principal submatrix of A on rows and columns $\{x, y, z, 1, 2, 3\}$, like A itself, has rank at most 3. Suppose that we had $d_y = 0$ while $d_z \neq 0$. Then the 4×4 submatrix on rows $\{x, y, z, 1\}$ and columns $\{y, z, 1, 2\}$ would be combinatorially nonsingular (that is, permutation equivalent to an upper-triangular matrix with nonzero entries on the diagonal), contradicting that $\text{rank}(A) \leq 3$. By the symmetries of G_6 , we could have chosen any pair instead of $d_y = 0, d_z \neq 0$, and thus if any one of the three quantities d_x, d_y , or d_z is equal to zero, then all three must be. But if all three of the first diagonal entries were zero, then the 4×4 principal submatrix on rows and columns $\{x, y, 1, 2\}$ would be combinatorially nonsingular. It follows that in the 10×10 rank-3 matrix A , the first three diagonal entries d_x, d_y , and d_z are all nonzero.

Considering once more the full matrix A , let $\beta = a_{x2}/a_{z2}$ and $\gamma = a_{y1}/a_{z1}$, so the first three rows of A can be written:

$$\begin{bmatrix} d_x & 0 & 0 & 0 & \beta a_{z2} & a_{x3} & 0 & a_{x5} & a_{x7} & a_{x8} \\ 0 & d_y & 0 & \gamma a_{z1} & 0 & a_{y3} & a_{y4} & 0 & a_{y7} & a_{y8} \\ 0 & 0 & d_z & a_{z1} & a_{z2} & 0 & a_{z4} & a_{z5} & a_{z7} & a_{z8} \end{bmatrix}.$$

Since d_x, d_y , and d_z are nonzero and A has rank at most 3, every other row of A can be obtained from these first three rows by taking a linear combination, and the coefficients of the linear combination are determined by entries in the first three columns. Every entry of A is thus determined by the variables appearing in the 3×10 matrix above. For any i and j in the set $\{1, 2, 3, 4, 5, 7, 8\}$, we have

$$a_{ij} = \frac{\bar{a}_{xi}}{d_x} a_{xj} + \frac{\bar{a}_{yi}}{d_y} a_{yj} + \frac{\bar{a}_{zi}}{d_z} a_{zj}.$$

In those cases where $i \neq j$ and ij is not an edge of G_{10} , the entry $a_{ij} = 0$ gives an equation on the entries of the first three rows. Using several such equations, we deduce that $d_x d_y > 0$, as follows:

1. The entries $a_{27} = 0$ and $a_{18} = 0$ give us the pair of equations

$$d_x a_{z7} = -d_z \bar{\beta} a_{x7} \text{ and } d_y a_{z8} = -d_z \bar{\gamma} a_{y8}.$$

2. Combining the equations from $a_{37} = 0$ and $a_{38} = 0$, we have

$$a_{x7}a_{y8} = a_{y7}a_{x8}.$$

3. Combining the equations from $a_{52} = 0$ and $a_{58} = 0$, we have

$$a_{x8} = \beta a_{z8}.$$

4. Combining the equations from $a_{41} = 0$ and $a_{47} = 0$, we have

$$a_{y7} = \gamma a_{z7}.$$

Multiplying the first pair of equations and then substituting in each of the remaining equations in order, then canceling the nonzero term $a_{z7}a_{z8}$, we arrive finally at

$$d_x d_y = d_z^2 \beta \bar{\beta} \gamma \bar{\gamma},$$

a positive quantity. This proves that, in any Hermitian matrix $A \in \mathcal{H}(G_{10})$ of rank no more than 3, the first two diagonal entries are nonzero and have the same sign. \square

Proof of Theorem 7.3. We review the definition of the graph G_{12} that will provide the claimed example: starting from the diagram of the cube with a labeled vector for every pair of faces, edges, or corners, we omit the vector 10 and connect pairs of vertices from the set $\{x, y, z, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ whenever their corresponding vectors are not orthogonal.

Letting M_{12} be the submatrix of M_{13} obtained by deleting the last column (labeled 10), we have $M_{12}^T M_{12} \in \mathcal{S}(G_{12})$ and thus $(3, 0) \in \mathcal{I}(G_{12})$ and $(3, 0) \in \text{h}\mathcal{I}(G_{12})$. It follows from Theorem 7.1 that the point $(2, 2)$ also belongs to $\mathcal{I}(G_{12})$ and $\text{h}\mathcal{I}(G_{12})$. To show that $\pi(G_{12}) = \text{h}\pi(G_{12}) = \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$, it suffices by the Northeast Lemma and symmetry to show that $(2, 1) \notin \text{h}\mathcal{I}(G_{12})$.

Let A be any matrix of rank 3 in $\mathcal{H}(G_{12})$, and let d_x , d_y , and d_z be the first three diagonal entries of A . Omitting rows and columns 6 and 9 gives us

a matrix of rank no more than 3 in $\mathcal{H}(G_{10})$, and so Lemma 7.1 tells us that d_x and d_y are nonzero and have the same sign. However, the automorphism group of G_{12} inherits all the symmetries of a cube with one marked corner, and thus anything true of the pair of vertices $\{x, y\}$ is also true of the pair $\{y, z\}$, so d_y and d_z are also nonzero and have the same sign. More explicitly, using the symmetry of a counterclockwise rotation of the cube around the corner marked 10, we delete rows and columns 4 and 7 (instead of 6 and 9) and reorder the remaining rows and columns as $(y, z, x, 2, 3, 1, 5, 6, 8, 9)$ to yield a different matrix belonging to $\mathcal{H}(G_{10})$, and invoke Lemma 7.1 again to obtain $d_y d_z > 0$, showing that the three diagonal entries d_x , d_y , and d_z all have the same sign.

The principal submatrix of A on rows and columns $\{x, y, z\}$ has either three positive eigenvalues or three negative eigenvalues, and so by interlacing the partial inertia of A must be either $(3, 0)$ or $(0, 3)$. This completes the proof that the graph G_{12} achieves the inertial partition and Hermitian inertial partition $(3, 3, 2)$, and is not inertia-balanced. \square

Of course the same argument also shows that $\pi(G_{13}) = \text{h}\pi(G_{13}) = \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$, but we are interested in the smallest possible graph that is not inertia-balanced. The following proposition justifies our claim that G_{12} is at least locally optimal.

Theorem 7.4. *Every proper induced subgraph of G_{13} is either isomorphic to G_{12} or is inertia-balanced and Hermitian inertia-balanced.*

Proof. By Theorem 7.1 every graph G with $\text{mr}(G) < 3$ is inertia-balanced and every graph G with $\text{hmr}(G) < 3$ is Hermitian inertia-balanced. It thus suffices to show that for every proper induced subgraph F of G_{13} other than G_{12} , $(2, 1) \in \mathcal{I}(F)$ unless $|F| < 3$.

Recall that G_{13} is defined by orthogonality relations between the columns

of the matrix

$$M_{13} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 1 & 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & -1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & -1 & 1 & 0 & -1 & -1 & 1 & 1 \end{bmatrix},$$

corresponding to various axes of symmetry of a cube. In other words, $M_{13}^T I_3 M_{13} \in \mathcal{S}(G_{13})$ (where the identity matrix I_3 imposes the standard positive definite inner product on \mathbb{R}^3) and so $(3, 0) \in \mathcal{I}(G_{13})$.

The automorphism group of G_{13} has three orbits, corresponding to the faces (x , y , and z), edges (1, 2, 3, 4, 5, and 6), and corners (7, 8, 9, and 10) of the cube. The deletion of any corner yields G_{12} (perhaps with a different labeling) and there is, up to isomorphism, only one way to delete two corners. Every proper induced subgraph of G_{13} other than G_{12} is thus isomorphic to an induced subgraph of $G_{13} - x$, of $G_{13} - 3$, or of $G_{13} - \{7, 8\}$. Letting G be each of these three graphs in turn, we exhibit for each a diagonal matrix D with $\text{pin}(D) = (2, 1)$ and a real matrix M such that $M^T D M \in \mathcal{S}(G)$.

$G = G_{13} - x$, $(2, 1) \in \mathcal{I}(G)$:

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad M = \begin{bmatrix} . & 0 & 2 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & 1 & 0 & 1 \\ . & 1 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & -2 & 3 & 2 & -3 \\ . & 0 & 3 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$G = G_{13} - 3$, $(2, 1) \in \mathcal{I}(G)$:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & . & 0 & 1 & -1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 0 & 2 & 0 & . & 1 & 0 & 1 & 4 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & . & 2 & 2 & 0 & 2 & 2 & 2 & 1 \end{bmatrix}$$

$G = G_{13} - \{7, 8\}$, $(2, 1) \in \mathcal{I}(G)$:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 1 & 0 & 1 & -1 & . & . & 1 & 2 \\ 0 & 1 & 0 & 2 & 0 & 1 & 1 & 0 & 1 & . & . & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 0 & . & . & 2 & 1 \end{bmatrix}$$

For each value of M and D , the matrix M^TDM has partial inertia $(2, 1)$ and belongs to $\mathcal{S}(G)$ for the desired subgraph G . If F is a proper induced subgraph of G_{13} other than G_{12} , then F is an induced subgraph of one of these three graphs. Part (a) of Proposition 4.2 allows us to delete vertices from any one of the three graphs and keep the partial inertia $(2, 1)$ as long as at least 3 vertices remain, which gives us $(2, 1) \in \mathcal{I}(F)$ unless $|F| < 3$. \square

Question 5. Is G_{12} the unique graph on fewer than 13 vertices that is not inertia-balanced?

Theorems 7.1 and 7.3 only permit us to answer the Inertial Partition Classification Problem and Hermitian Inertial Partition Classification Problem up to height 3. We have shown examples of constructing a graph whose minimum rank realization with a particular partial inertia is sufficiently “rigid” to prevent intermediate partial inertias of the same rank between the matrix and its negative. If the rank is allowed to increase, though, it is much less clear what restrictions can be made. The next difficult question appears to be whether $(4, 4, 4, 3) = \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{smallmatrix}$ is an inertial partition.

Question 6. Let G be a graph and let M be a matrix in $\mathcal{S}(G)$ with $\text{pin}(M) = (4, 0)$. Must there exist a matrix $M' \in \mathcal{S}(G)$ with $\text{pin}(M') = (3, 2)$?

On the one hand, partial inertia $(3, 2)$ is of higher rank than partial inertia $(4, 0)$, which means that any proof along the lines of Theorem 7.3—a proof that a particular arrangement of orthogonality relations of vectors in \mathbb{R}^4 could not be duplicated in \mathbb{R}^5 with an indefinite inner product of signature $(3, 2)$ —would have an extra degree of freedom to contend with. On the other hand, there seems to be little hope of constructing the matrix M' directly from M using any sort of continuous map such as that employed in the proof of Theorem 7.1.

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